Sequential Bargaining with Multiple Buyers

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Abstract

A seller bargains with two buyers to make a deal with each of them, using an alternating-offer protocol (“AO”). The bargaining begins with one buyer, with the second entering at a future date. The seller has a concave utility function defined over total payments from buyers, so the two bargains affect each other. When the seller’s utility function exhibits decreasing absolute risk aversion, a higher price in the first bargain increases the price in the subsequent bargain. Even if two players are identical except for the arrival date, they will make different payments to the seller. The shape of the utility and the arrival date determine whether there is a first or second-mover advantage. Although agreements in our model are reached on different dates, the usual limit payoffs for AO do not approach those of the sequential Nash bargaining solution. Finally, we extend the model to a vertical market, in which an upstream seller supplies downstream buyers with critical input. These buyers compete with each other in the downstream market. We find that, even if the buyers are symmetric Cournot competitors, the equilibrium of the model is asymmetric, with one buyer paying more than the other. Prior to entry by the second firm, the price set by the incumbent can decrease with the increased expected entry dates. Standard vertical models would not predict this.

Keywords: Sequential bargaining, multilateral bargaining, concave utility

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1 Introduction

The literature on alternating offer ('AO') bargaining model (Rubinstein, 1982) has said little about the behavior at the market level. If we want to use AO bargaining model to say something about how bargaining affects market outcomes, we need to account for the fact that the bargain made by one buyer with a seller may affect the profits of another buyer who has made her own bargain with the seller. In this setup, it is clear that bargaining theory must consider the interdependence between bargains if it is to be useful in elucidating market behavior. Therefore, we have been obliged to extend the AO bargaining model to account for interdependent bargains.

There are only a few papers that study multiple episodes of bargaining, in which one side is bargaining with multiple partners to make multiple interdependent deals. These papers use the Nash bargaining solution ('NBS') (Horn and Wolinsky, 1988). Or they introduce extra interaction between buyers to induce the interdependence between bargains (Abreu and Manea, 2023). The analysis focusing on the pure interaction between bargains without the interaction between buyers under the solution concept of Rubinstein (1982) is still missing.

The starting point of our paper is that we study the interaction between bargains. We analyze a bargaining model with one seller and two buyers that is different from Horn and Wolinsky (1988) in that we based ours on the solution concept of Rubinstein (1982).

For most of our analysis, interdependence between bargains comes from our assumption that the seller has a concave utility function defined on the total payment by buyers. For the sake of simplicity, we assume that buyers have linear utility.

In our model, the seller and one buyer are in the game at the beginning, while the other buyer arrives at a known future date. The seller bargains for a contract with each of the two buyers. The contract generates a flow payoff 1 to the buyer each period, and buyers and the seller bargain over the flow price paid to the seller each period.

Before the arrival of the second buyer, there is a two-player AO bargaining game between the seller and the first buyer. If they reach an agreement before the second buyer arrives, there will be another two-player AO bargaining game between the seller and the second buyer. If the first buyer does not have an agreement when the second buyer arrives, the seller will bargain with two buyers simultaneously in an AO bargaining protocol. The seller makes offers to both buyers simultaneously in odd periods and two buyers make offers to the seller simultaneously in even periods.

As an application, we consider the situation of two health insurance providers who want to include a hospital in their networks. To do that, insurers need to pay a fixed transfer to the hospital. The hospital will provide treatment to patients who buy insurance from one of the two insurers. In this example, the hospital is the seller, and the insurers are the buyers. The hospital can be included in both networks and will bargain with two insurers to decide on the transfers. Two negotiations can start on different dates. Alternatively, the hospital may be renewing contracts with two insurers, and those contracts have different termination dates.

To motivate the hospital's concave utility in monetary payments, we can consider that the hospital uses these payments to fund other projects. Ranking them by profitability, the hospital will start from the project with the highest return and work its way down, as long as it has the funds to do so. Thus, the marginal payoff from the monetary payments will decrease. This will generate a concave seller utility function defined over total payments to the seller.

Given concave seller utility, if the seller already has a payment from the first bargain, she is less eager for the extra payment from a second bargain. This assumption leads to the interaction between the bargains with the first buyer and the second buyer. If we assume linear utility for the seller, then the two bargains do not affect each other. The outcome of our model is simply two separate Rubinstein bargains.

If the first buyer has already reached an agreement with the seller when the second buyer arrives, the second flow price is affected by the first flow price. The specific form of the utility function determines how the second flow price is affected.

As we will see, we can think of seller utility from rejecting as if it were a special kind of lottery. Risk preferences in this lottery drive the relationship between bargains.

- When the seller has decreasing absolute risk aversion (DARA), the second price increases with the first price.

- When the seller has increasing absolute risk aversion (IARA), the second price decreases with the first price.

If the first buyer does not reach an agreement with the seller until the second buyer arrives, the seller will propose a price to both buyers simultaneously in her turns and the buyers will propose simultaneously to the seller in their turns. The equilibrium in this continuation game is symmetric, where buyers pay equally.

Based on the results above, we construct the equilibrium for the whole game, finding that two agreements are reached immediately after each buyer arrives. In this equilibrium, though two buyers are identical except for the arrival dates, they end up at different prices. The shape of the utility function and, also, the arrival date T determine whether there is a first-mover advantage or second-mover advantage in sequential bargaining.

Comparing the equilibrium to a one-seller-one-buyer Rubinstein bargaining with the same utility function, we find having a second buyer can benefit the first buyer in some situations and harm her in others. For example, with DARA, the first buyer can benefit from the existence of the second buyer. However, the second buyer is always worse off, compared to the case where she is the only buyer.

We also consider limit results of our AO bargaining model, where the intervals between two proposals become arbitrarily small, and the arrival time between buyers remains unchanged. The limit result is asymmetric as in our model, but it is not the solution to a sequential Nash bargaining problem. If we consider a simpler limit result where the discount factor approaches one, the limit outcomes in both bargains are the same as in simultaneous Nash bargaining. However, even with the discount factor going to one, the solution to our sequential AO bargaining model does not go to *sequential* NBS. In other words, the relationship between the two bargains is different from that generated by sequential Nash bargaining.

We extend our analysis to the case in which the seller is bargaining with two buyers, both of whom compete with each other in a downstream market. This extends the setup in Horn and Wolinsky (1988) to AO bargaining. The first buyer is an incumbent monopolist. The other one arrives at a known date. If input prices were determined on a take-it-or-leave-it ('TIOLI') basis, the incumbent's price would not change with the date when entry occurs.

In our model, the incumbent's price is affected by the entry date. One possible outcome is: the further in the future is the entry date, the lower is the output price in the pre-entry period. Another difference concerns market shares. In the TIOLI setup, if the two buyers are otherwise identical, they pay the same price for the input, and have equal shares of the downstream market. Our model generates different input prices for the two buyers, and, hence, different downstream market shares.

Related literature. Though there is a large literature on bargaining, most papers study single-episode bargaining problems, where players bargain to split one pie. They have studied this problem with many variations, such as incomplete information (Fudenberg and Tirole, 1983; Cramton, 1992), and different bargaining protocols (Baron and Ferejohn, 1989; Cho, 1990).

There are only a few papers studying multiple-episode bargaining problems. For example, Horn and Wolinsky (1988) studied a monopoly supplier of an input bargaining with two firms to sell inputs to both firms while firms compete in a downstream market. In their model, the supplier is bargaining over input prices with two downstream firms. Their bargains interact because there is downstream competition between firms. Horn and Wolinsky used the sequential NBS to characterize the bargaining.

A more recent paper, Abreu and Manea (2023), studies the situation where a seller sells products to a group of buyers using the random-proposer bargaining protocol. Players have linear utilities, and the interaction between the bargains with different buyers comes from scarce capacity. The seller does not have enough output to satisfy the demand of all buyers and there will be competition among buyers for outputs.

Unlike the above two papers, we use the solution concept of Rubinstein (1982) to study the multiple-episode bargaining problem, and the interaction across episodes does not come from interactions between buyers as in Abreu and Manea (2023). Instead, the interaction comes from the concave utility function of the seller. It reveals that two bargains do not require buyers to interact to affect each other. A common assumption of concave utility can lead to the interaction. As noticed above, the concavity may reflect differing investment options available to the seller.

The concave utility function in bargaining has also been studied in the literature. Hoel

(1986) studies the nonlinear utility in the bargaining problem and finds that the limit price of AO bargaining goes to the Nash bargaining price as players become patient. Sobel (1981) considers a concave, increasing von Neumann-Morgenstern utility function for players and establishes the relationship between a class of bargaining solutions, like those of Nash and Raiffa, Kalai. Crawford and Varian (1979) finds that even if players are allowed to misrepresent their utilities as a weakly concave function in the Nash bargaining problem, they would report linear utility functions. Volij and Winter (2002) study the risk aversion in the bargaining game, which is a property of the concave utility function. White (2008) studies the effect of prudence in a bargaining problem with risk. Adding to the literature, our model studies how risk aversion affects the interaction between two bargains instead of within one bargain.

The organization of the paper is as follows. Section 2 provides the setup. Section 3 introduces the main message about the interaction between bargains. Section 4 discusses an extension of the model. Finally, Section 5 concludes the paper.

2 Model

We study a bargaining model with three players, one seller, and two buyers (B_1, B_2). The seller wants to make two contracts and each buyer demands one contract. The seller bargains with each buyer for one non-changeable contract.

The time is discrete, and the horizon is infinite ($t = 1, 2, ...$). The seller and B_1 arrive at $t = 1$ and start bargaining, while B_2 arrives at date $T > 1$. Here we assume T to be odd to ensure that the seller is still the first proposer when B_2 arrives, but it is not essential. In other words, departing from the model in Rubinstein (1982), we have a second buyer arriving at date T and starting to bargain with the seller thereafter.

When the seller and the buyer reach an agreement, the contract between them will take effect immediately and last permanently. The buyer gets a flow payoff of 1 from the contract each period, and she also pays a flow price of x to the seller each period, which is determined by bargaining. Thus, the flow monetary payoffs are x to the seller and $1 - x$ to the buyer. We assume that the seller has a concave utility function $u(.)$ over a total payment of x in a period. The utility $u(.)$ is (1) strictly increasing, (2) twice differentiable, and (3) strictly concave. On the contrary, buyers have a linear utility $v(x) = x$.[1] Suppose the flow prices in two bargains are x_1 and x_2 respectively. When only one bargain is made, the seller's utility for each period is $u(x_1)$; when both bargains are made, the seller's utility for each period is $u(x_1 + x_2)$. As for buyers, they have flow utility $1 - x_1$ and $1 - x_2$.

2.1 Bargaining protocol

Players use alternating-offer (AO) bargaining to determine the flow payment in each contract, which is similar to Rubinstein (1982). In odd periods, the seller proposes a flow price to every active buyer, and each active buyer decides to accept or reject; in even periods, each active buyer proposes to the seller, and the seller decides to accept or reject. Once an offer is accepted, the contract takes effect immediately, and the buyer leaves the game (becomes inactive).

Before T, there is only one buyer in the game, which is a one-to-one alternating offer bargaining. After T, there are two possibilities. If B_1 has left the game, the bargain between the seller and B_2 is the same as Rubinstein bargaining, apart from the concavity of the seller's payoff. If buyer 1 has not left the game, then the seller bargains simultaneously with the two buyers. The seller proposes offers to both buyers simultaneously in her turns, and two buyers propose offers to the seller simultaneously in their turns.

2.2 Strategies and equilibrium concept

We assume that the seller observes the whole history of the game. This includes all past offers, acceptances, and whether buyers are active in the game. However, a buyer can only observe her own bargaining history. The buyer cannot observe the detailed offers made in the other bargain and can only observe the date of agreement.[2] Therefore, this is a game of complete, but imperfect information.

[1]Our results can be generalized to a strictly increasing and strictly concave buyer utility $v(x)$. By letting $\hat{v} = v(x)$, we can write the seller's payoff as $u(v^{-1}(\hat{v}))$, and the buyer's payoff as $\hat{v}$. Notice that if $v(.)$ is concave, the function $u(v^{-1}(\hat{v}))$ is a decreasing and concave function in $\hat{v}$, so we can transform the problem of two concave utility functions into the problem of one concave utility function and one linear utility function.

[2]If the other buyer has not made an agreement, the buyer knows that she is still in the game.

As for the seller's strategy, in an odd period, she proposes a flow price $x \in [0,1]$ to each active buyer in the game, conditional on the whole history. In an even period, she sees the proposed flow price by each active buyer and the past history and decides whether to accept the offer from each active buyer.

As for the buyers' strategy, in an even period, active buyers will propose flow prices $y \in [0,1]$ to the seller based on the history that each buyer observes. In an odd period, an active buyer will decide to accept an offer from the seller, given the offer from the seller and the history observable to that buyer.

In a period, if there are two contracts that have been made, and B_1 pays a flow price x_1 and B_2 pays a flow price x_2, then the normalized flow payoff of the seller is $(1-\delta)u(x_1+x_2)$, while two buyers have the flow payoff $(1-\delta)(1-x_1)$ and $(1-\delta)(1-x_2)$. Here $1-\delta$ is scaling the payoff.

We use perfect Bayesian equilibrium (PBE) as the equilibrium concept. In this equilibrium, at each information set, players maximize the discounted sum of normalized flow payoffs. And we apply the refinement of *passive beliefs* for buyers.

Passive beliefs refer to B_i's belief about what is happening with B_j when B_i receives an out-of-equilibrium offer from the seller in the current period, which affects her continuation payoff. *Passive beliefs* means that B_i will believe the seller is playing on-path strategy to B_j in the current period, even if B_i observes an out-of-equilibrium offer from the seller. Moreover, with passive beliefs, if an agreement is reached before date T, B_2's belief about the first price does not change with the seller's actions in the bargaining with B_2. The belief only depends on the agreement date B_2 observes.

Additionally, we need to assign the off-path belief to buyer 2 when he observes an off-path agreement date. With some belief assignments, we can have an equilibrium where the agreement is not reached immediately. However, We can use a refinement similar to D1 criterion to exclude this probability (it is something in the spirit of D1 criterion since the first bargaining is a game between seller and buyer 1 instead of a simple signaling).

We are considering the belief of buyer 2 when the equilibrium agreement date is $t > 1$.

To look at the off-path belief observing the agreement date reached at period 1, define

$$D(p_1,[0,1]) := \bigcup_{\mu\in[0,1]} \{p_2 = \hat{x}_2(\mu) : U_1(p_1,p_2) > U^* \textit{ and } p_1 \leqslant 1-\delta^t+\delta^t p^t\}$$

$$D^0(p_1,[0,1]) := \bigcup_{\mu\in[0,1]} \{p_2 = \hat{x}_2(\mu) : U_1(p_1,p_2) \geqslant U^* \textit{ and } p_1 \leqslant 1-\delta^t+\delta^t p^t\}$$

where p_1 is the price proposed by the seller in period 1, μ is buyer 2's belief of the price in bargain 1 seeing agreement date period 1, p^t is the equilibrium proposal in period t, U^* is the seller's equilibrium payoff, and $U_1(p_1,p_2)$ is the seller's payoff where the seller proposes p_1 in period 1 and the price in the bargain 2 is p_2.

These are the sets of prices in the second bargain such that buyer 1 accepts the offer p_1 in period 1 and the seller is strictly/weakly better off by proposing p_1 in period 1 than in the equilibrium. Notice that when $p_1 > 1-\delta^t+\delta^t p^t$, two sets are always empty. A first-bargain price p is eliminated by D1 criterion if there is another price p' such that $D^0(p,[0,1]) \subset D(p',[0,1])$. With this refinement on buyer 2's off-path belief seeing an agreement reached at date 1, we can exclude the possibilities of delay in the agreement.

3 Equilibrium behavior in sequential bargaining

To solve for the equilibrium of the whole game, we first need to figure out what happens in the bargaining between the seller and B_2 if the seller already has an agreement with B_1 before the date T. We also need to figure out what will happen if there is no agreement before the date T.

3.1 Bargaining after having an agreement with B_1

In this subsection, we analyze what would happen in the second bargain, if the seller and B_1 have already reached an agreement before B_2 arrives.

The equilibrium of the continuation game can be constructed in the same way as Rubinstein (1982). Let the seller propose the price x_2 in her turns, and B_2 proposes the price y in her turns. They satisfy

$$\begin{cases} 1 - x_2 = \delta(1-y) \\ u(x_1+y) = (1-\delta)u(x_1) + \delta u(x_1+x_2) \end{cases} \tag{1}$$

The first equation requires that the seller's proposal makes B_2 indifferent between accepting and rejecting. The second equation requires that the buyer's proposal makes the seller indifferent between accepting and rejecting. From (1), we can solve for x_2 as a function of x_1, denoted as $\hat{x}_2(x_1)$. This is the price that the second bargain will arrive at, given the price reached in the first bargain.

If the seller has an agreement with buyer 1 at price x_1, the unique equilibrium outcome of the continuation game starting at T is that the seller proposes a price following a function $\hat{x}_2(x_1)$ and B_2 accepts at date T, where $\hat{x}_2(x_1)$ is determined by (1).

Uniqueness can be proved using an argument similar to Shaked and Sutton (1984). See Appendix A.1.

Given a payment in the first bargain, this will affect the outcome of the second bargain. The direction of the effect of x on $\hat{x}_2(x)$ depends on the specific form of the utility $u(.)$. We address this in Proposition 1.

Proposition 1

- *If $u(.)$ has decreasing absolute risk aversion (DARA) , $\hat{x}_2(.)$ is increasing*
- *If $u(.)$ has increasing absolute risk aversion (IARA) , $\hat{x}_2(.)$ is decreasing*

To see this, we first rewrite (1) as

$$u\Big(x_1 + 1 - \frac{1-\hat{x}_2(x_1)}{\delta}\Big) = \delta u\big(x_1 + \hat{x}_2(x_1)\big) + (1-\delta)u(x_1) \tag{2}$$

The RHS of (2) is the seller's payoff from rejecting B_2's offer, which can be regarded as the expected payoff of a lottery. This lottery has the probability of δ that the seller gets $x_1 + \hat{x}_2(x_1)$, and a probability of $1-\delta$ that she gets x_1. Thus, the LHS, which is the buyer's proposal, is the certainty equivalent of the lottery. Figure 1 illustrates the expected payoff

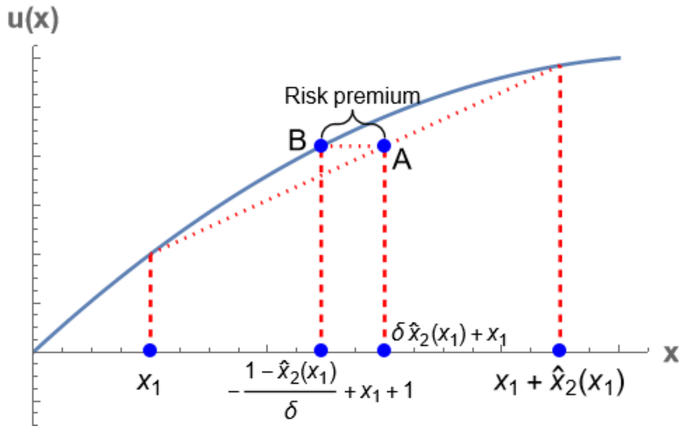


Figure 1: How risk aversion affects the second price

of the lottery (point A) and its certainty equivalent (point B), as well as the risk premium. At point B, the certainty equivalent payoff of the seller is given by $-\frac{1-\hat{x}_2(x_1)}{\delta}+x_1+1$

If the seller is more risk-averse, she has a larger risk premium and thus a smaller certainty equivalent. As a result, B_2 will propose less to her, which leads to a smaller flow price $\hat{x}_2(x_1)$.

Consider a utility $u(.)$ with DARA. If x_1 is larger, then the seller is more wealthy. According to DARA, the seller is less risk averse with a larger x_1. Thus, as x_1 goes up, the certainty equivalent of the lottery, which is also the buyer's proposal, becomes larger, which leads to a larger $\hat{x}_2(x_1)$. Similarly, with IARA, $\hat{x}_2(x_1)$ decreases with x_1.

On the other hand, we can regard the cost of delay as the probability of breaking down instead of the discount factor. Having a discount factor of δ is the same as having a breakdown probability $1-\delta$, because both of them make the rejection payoff be $(1-\delta)u(x_1)+\delta u\big(x_1+\hat{x}_2(x_1)\big)$. In this explanation, a seller who is less risk-averse is less afraid of the possible breakdown and can ask more from the buyer.

3.2 Simultaneous alternating offer bargaining with two buyers

In this subsection, we introduce what would happen if there is no agreement made before the date T. This would set up a simultaneous bargaining at that point.

The two buyers can make or receive different offers. However, in the equilibrium of this continuation game, the outcomes are symmetric: buyers pay equally to the seller.

Suppose, in equilibrium, the seller proposes x^* to both buyers. If a buyer rejects, she believes that the other buyer will accept the offer x^*, so she will bargain with a seller who has a contract at the price x^* starting next period. As a result, in the next period, the buyer who rejects proposes $1 - \frac{1-\hat{x}_2(x^*)}{\delta}$ in her turn. Then, x^* should make the buyer indifferent between accepting and proposing in the next period:

$$1 - x^* = \delta(1 - (1 - \frac{1 - \hat{x}_2(x^*)}{\delta})) \Leftrightarrow x^* = \hat{x}_2(x^*)$$

If the seller does not have an agreement with buyer 1 by date T, the equilibrium outcome of the subgame starting at T is that the seller proposes x^* to both buyers and buyers accept at date T, where x^* solves $x = \hat{x}_2(x)$.

Notice that though any equilibrium has the condition $x = \hat{x}_2(x)$, the uniqueness is not guaranteed since $\hat{x}_2(x)$ may have more than one fixed point.

With IARA, $\hat{x}_2(x)$ is decreasing according to Proposition 1. In this case, $\hat{x}_2(x) = x$ has only one solution.

However, with DARA, $\hat{x}_2(x)$ is increasing according to Proposition 1, and $\hat{x}_2(x) = x$ may have more than one solution. One sufficient condition to guarantee the uniqueness of the fixed point is $\hat{x}_2'(x)$ being monotone.

Notice that the multiplicity here does not lead to a delay in the equilibrium because the buyer cannot observe the offer the other buyer receives in the current period. With multiple equilibria, one possible way to construct a delay in equilibrium is that in the current period, players are playing a bad equilibrium for buyers, but in the next period, they play a better equilibrium for buyers, so buyers may want to delay. However, with passive belief, no matter what offer the seller proposes to her, she will believe that the other buyer will accept the offer, and the game will not proceed to a better equilibrium for her if only one buyer rejects.

As for the possibility that both buyers reject in equilibrium and in the next period they go to a better equilibrium for buyers, the seller will want to propose a smaller price to have buyers accept the offers. By doing this, the seller is in the situation of the first proposer, while letting buyers reject makes buyers the first proposer. Besides, there is a discount cost in delay.

3.3 Equilibrium results

With these results, we can construct the equilibrium of our model. At date $T-1$, the seller knows that if she rejects, then, in the next period, she will get involved in simultaneous bargaining with two buyers. She will get a flow payoff $u(2x^*)$ from that. Considering this, B_1 will do backward induction accordingly. Further backward induction gives the seller's proposal at date $t = 1$, which will be accepted by B_1 immediately. Let us call this proposed flow price at date $t = 1$ in equilibrium as x_1^*. Then, the second bargain generates an equilibrium flow price of $x_2^* = \hat{x}_2(x_1^*)$, and the agreement is done at date T.

Proposition 2 *There is an equilibrium outcome of the game where two agreements are reached immediately at date* 1 *and date* T*.*

Notice that the process of backward induction does not cause multiplicity, so whether we have a unique equilibrium in the game depends on whether we have a unique equilibrium in the simultaneous bargaining.

The bargaining process with two active buyers has a unique result when $\hat{x}_2(x)$ has a unique fixed point. The result of backward induction is also unique, so our x_1^* is unique in this case. Moreover, the outcome is unique in the continuation game after the seller and B_1 reach an agreement of x_1^*. Therefore, the outcome that B_1 pays a flow price of x_1^* and B_2 pays a flow price of $x_2^* = \hat{x}_2(x_1^*)$ is the unique equilibrium outcome in our model when $\hat{x}_2(x)$ has a unique fixed point.

The possible multiplicity of the equilibrium also draws attention to the current structural model used in IO research. As noted, empirical IO researchers often assume risk-neutral simultaneous NBS. However, there is no reason to assume that risk neutrality is correct. Failure to recognize the possibility of concave utility for the seller may make researchers overlook other possible equilibrium outcomes.

3.4 How bargains affect each other

To see how these two episodes of bargaining affect each other, we compare the equilibrium flow prices in our model (x_1^*, x_2^*) to the one-seller-one-buyer Rubinstein bargaining with the same utility. If the results are different, bargains are affecting each other since adding a second buyer to the game changes the equilibrium outcome. We refer to the one-seller-one-buyer Rubinstein bargaining with the same utility as the *unaffected bargaining* and denote its flow price outcome as x_U. In the application of the hospital and insurers, this comparison is comparing the situations of the insurer being the only buyer of the hospital's services and having a second buyer.

When $u(.)$ is concave, because x_1^* and x_2^* are not always equal to the unaffected bargaining price x_U, the existence of one bargain affects the other bargain. The first flow price affects the second flow price depending on the form of $u(.)$. Players in the first bargain will change their actions strategically due to the existence of such an effect. On the contrary, if $u(.)$ is linear, such an interaction between two bargains does not exist: both flow prices are the same as the unaffected bargaining. Therefore, the interaction comes from the concavity (risk aversion) of $u(.)$.

The relationship between x_1^*, x_2^*, and x_U is ambiguous. It depends on the specific form of the utility function $u(.)$ and T.

If $u(.)$ has DARA, then $x_2^* > x_U$. If $u(.)$ has IARA, then $x_2^* < x_U$. To see this, notice that the outcome from the unaffected bargaining is the same as the bargaining between the buyer and the seller with no previous agreement, $x_U = \hat{x}_2(0)$. The seller is less wealthy without a previous agreement, so with DARA, the seller is more risk averse in this situation, and the price she can get is smaller. For IARA, it is the opposite way.

As for the relationship between x_1^* and x_U, assuming DARA does not give a consistent result. For example, assume a utility with DARA, $u(x) = \sqrt{x}$ and $\delta = 0.9$, we have $x_1^* > x_U$ with $T = 3$, and $x_1^* < x_U$ with $T = 9$.

If we assume IARA for the seller, we have $x_1^* < x_U$.

Compared to the situation where there is only one buyer, she is better off if she has another buyer in the game if the seller has IARA. With the more realistic assumption of DARA, being the only buyer is better than being the second one in the sequential bargaining. However, when comparing with being the first one in sequential bargaining, whether being the only buyer is better off or worse off is ambiguous. The first buyer knows that her payment also changes the seller's payment in the second bargaining process. The strategic interaction in the bargaining can make the first buyer pay more or less than if she were the only buyer.

3.4.1 Forces affecting bargaining outcome

In this section, we examine the economic forces behind the setting of the payments made by the first and second bargainers. The setting of the first price is complicated. This is because it is affected by two forces. One force is the absolute risk aversion of the seller. The second force is the marginal utility of the second price, which affects the marginal utility of the first price. We have discussed how the first force affects the second price x_2^*. However, the first price x_1^* is affected by both forces.

To see why x_1^* is affected by two forces, recall that x_1^* is determined by backward induction. Let the seller's proposal in periods 1 and 3 be x^3, and the buyer's proposal in period 2 be y^2. Using backward induction from periods 2 to 1, we have:

$$1 - x^1 = \delta(1 - y^2)$$

Backward induction from period 3 to 2 yields:

$$(1-\delta^{T-2})u(y^2) + \delta^{T-2}u\Big(y^2 + \hat{x}_2(y^2)\Big) = (\delta - \delta^{T-2})u(x^3) + \delta^{T-2}u\Big(x^3 + \hat{x}_2(x^3)\Big)$$

Thus, we can link periods 3 and 1. We have the following.

$$\begin{aligned} &u(1 - \frac{1-x^1}{\delta}) + \delta^{T-2}\Big[u\Big(1 - \frac{1-x^1}{\delta} + \hat{x}_2(1 - \frac{1-x^1}{\delta})\Big) - u(1 - \frac{1-x^1}{\delta})\Big] = \\ &\underbrace{\delta u(x^3)}_{\text{Certainty equivalent}} + \delta^{T-2} \underbrace{\Big[u\Big(x^3 + \hat{x}_2(x^3)\Big) - u(x^3)\Big]}_{\text{Marginal utility from second payment}} \end{aligned} \tag{3}$$

The expression for x^1 illustrates two forces. The first terms on both sides of (3) represent the first force, and the second terms on both sides of (3) are the marginal utilities of the price in the second bargaining, representing the second force.

For the certainty equivalent effect, notice that the first term on the RHS of (3) can be regarded as the expected payoff of a lottery, which gives x^3 with probability δ, and 0 with probability $1 - \delta$. Consequently, the first term on the LHS is the certainty equivalent of this lottery. Thus, when the seller is less risk averse, she will have a lower risk premium and a higher certainty equivalent, which makes x^1 higher. This is how the certainty equivalent affects the x^1.

We also have terms on both sides, which represent the marginal utilities of the second payment. Adding the certainty equivalent effects to the marginal utility effects will determine the value of x^1.

For example, we first consider $\bar{x}^1$ that only equates only the certainty equivalence terms, i.e., the first terms on two sides, i.e., $u(1 - \frac{1-\bar{x}^1}{\delta}) = \delta u(x^3)$. Now account for the differing marginal utility effects, (adding these terms back into the equation). This gives the following. We are comparing

$$u(1 - \frac{1-\bar{x}^1}{\delta}) + \delta^{T-2}\Big[u\Big(1 - \frac{1-\bar{x}^1}{\delta} + \hat{x}_2(1 - \frac{1-\bar{x}^1}{\delta})\Big) - u(1 - \frac{1-\bar{x}^1}{\delta})\Big] \tag{4}$$

and

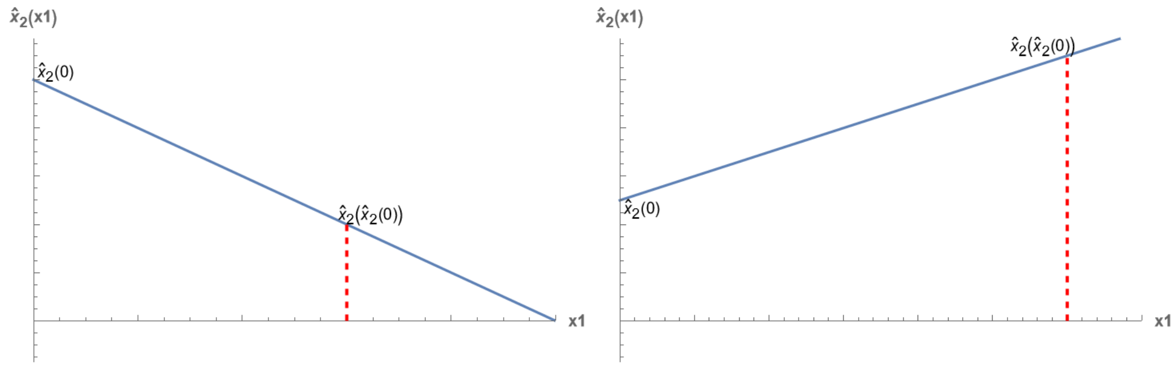
$$\delta u(x^3) + \delta^{T-2}\Big[u\Big(x^3 + \hat{x}_2(x^3)\Big) - u(x^3)\Big] \tag{5}$$

If the marginal utility of the second payment in (4) is smaller than in (5), that means $\bar{x}^1$ does not equate the equation (3), and is making LHS too small. Thus, x^1 that equates (3) is larger than $\bar{x}^1$. In this situation, the second force is making x^1 larger compared to the case without the second force. Similarly, if the marginal utility of the second payment in (4) is larger than in (5), the second force is making x^1 smaller.

As for whether the marginal utility of the second payment is larger in (4) or (5), it depends on how large the second prices are and the seller's marginal utility $u'(.)$.

First notice that for $\bar{x}^1$ s.t. $u(1 - \frac{1-\bar{x}^1}{\delta}) = \delta u(x^3)$, we have $1 - \frac{1-\bar{x}^1}{\delta} < x^3$ since $u(.)$ is an increasing function. With DARA, $\hat{x}_2(.)$ is increasing, so $\hat{x}_2(x^3) > \hat{x}_2(1 - \frac{1-\bar{x}^1}{\delta})$, which tends to make the marginal utility in (5) larger. However, larger x^3 not only makes the second price larger but also makes $u'(x^3)$ smaller due to concavity. Thus, The second force has an ambiguous effect with DARA. As for IARA, smaller $1 - \frac{1-\bar{x}^1}{\delta}$ is making both $\hat{x}_2(1 - \frac{1-\bar{x}^1}{\delta})$ and $u'(1 - \frac{1-\bar{x}^1}{\delta})$ larger, so the second force here makes x^1 smaller.

A more direct intuition of the second force is that when the marginal utility from the second price is large, the seller will require less first price from buyer 1 since she is satisfied



(a) Decreasing $\hat{x}_2(.)$

(b) Increasing $\hat{x}_2(.)$

Figure 2: Which buyer has advantage

more easily.

These forces can explain the relationship between x_1^*, x_2^*, and x_U. x_2^* is only affected by the first force, so DARA, which leads to a less risk-averse seller in bargain 2, will make $x_2^* > x_U$.

But x_1^* is affected by both forces, with DARA, and the second force may lead to a larger or smaller x_1^*, which leads to the ambiguous results in the comparison between x_1^* and x_U.

3.4.2 Bargaining strategy results

In our model, two identical buyers end up at different flow prices. It is natural to ask which buyer will pay less. The specific form of $u(.)$ and T determine which buyer pays a lower flow price.

To see the ambiguity here, look at the case where T is arbitrarily large. As T goes to infinity, the outcome of the first bargain goes closer and closer to the unaffected bargaining, because the effect of the second bargain in the far future is negligible. As a result, the first flow price will be close to $x_U = \hat{x}_2(0)$. This means that the second flow price must be close to $\hat{x}_2(\hat{x}_2(0))$. The shape of $\hat{x}_2(.)$ will affect the relationship between two flow prices. For example, as shown in Figure 2 (a), if $\hat{x}_2(.)$ is decreasing (IRAR), the first flow price ($x_1^* = \hat{x}_2(0)$) is higher than the second flow price ($x_2^* = \hat{x}_2(\hat{x}_2(0))$). Figure 2 (b) says that the first price is lower than the second price when $\hat{x}_2(x)$ is increasing (DARA).

We can see that in our bargaining setup, buyers do not always have a first-mover advantage. The seller's attitude towards risk will affect which buyer has an advantage. If the seller has IARA and $\hat{x}_2(.)$ is decreasing, there can be a second-mover disadvantage. If the seller has DARA and $\hat{x}_2(.)$ is increasing, there can be a first-mover advantage for buyers.

These two forces affecting the bargaining outcome can also explain the relationship between x_1^* and x_2^*. In our example of large T, the second force is negligible, since the second terms on both sides of 3 go to zero as T goes up. Therefore, the certainty equivalent effect dominates when T is large. This says that when the seller is less risk averse, she has a larger certainty equivalent and thus a larger flow price from bargaining. With DARA in Figure 2 (b), the seller is more wealthy and thus less risk-averse in bargaining 2, so the first force makes her x_2^* larger than x_1^*.

In another example, assume $u(x) = \sqrt{x}$, $\delta = 0.9$, we find that $x_1^* = 0.431$, $x_2^* = 0.487$ when $T = 3$, $x_1^* = 0.356$, $x_2^* = 0.483$ when $T = 11$. Notice that a smaller T is making x_1^* and x_2^* closer to each other. In this example, we have a utility function displaying DARA. Therefore, the certainty equivalent effect says that x_2^* should be larger, but the marginal utility effect can make x_1^* larger and thus make two prices closer. Notice that the second force has more effect with smaller T, so the prices in two bargains are closer to each other at $T = 3$ than $T = 11$.

Another interesting point arises when assuming that one buyer values the contract as 2 each period and one values the contract as 1 each period. Which buyer would the seller prefer to bargain with first? This is a question asked frequently in reality. For example, a hospital may need to bargain with a small insurance provider and a large insurance provider to decide the contracts to be included in the network. The hospital may want to bargain with the large provider first so she can get more patients quickly, or she can bargain with the small provider first to give her more market power, and thus the hospital can have more bargaining power in the bargain with the large provider. Schulman and Sibley (2023) discussed a similar problem under downstream interaction of insurance providers using the Nash bargaining solution, and they found that it is better to bargain with the small provider so that more competition in the downstream market makes the large provider worse off.

However, in our model, having the small buyer coming first does sometimes induce higher total payments from buyers. However, having the small buyer first makes the large payment from the large buyer come later, which increases the discount cost an can overturn the tendency to have the small one coming first. For example, when $\delta = 0.9$, $u(x) = \sqrt{x}$, $T = 3$, and two buyers value the contract 1 and 2 respectively, the total payment each period $(x_1^* + x_2^*)$ when the small one comes first is larger, but the present value of dealing with the big buyer first is higher.

Though we have an opposite result compared to Schulman and Sibley (2023), two results do not actually conflict. In the NBS method, there is no discount cost. In our model, sometimes having the small one first can make the total payment larger, which is consistent with their result. However, this benefit of having the smaller one first can be offset by the discount cost concerns.

3.5 Limit results

There is a tradition in the AO literature of proving that in the limiting cases, the AO result converges to the NBS result. We look at two limit results. The first limiting case is letting δ go to 1. The second is to fix T, but allow for an interval of size Δ between bargains. In this approach, we can look at the limiting result where the interval between proposals Δ goes to zero, but the arrival time between buyers T remains unchanged. In the limit, there are infinite periods before time T. Now we write the discount cost of one-period delay as $e^{-r\Delta}$, where r is the discount factor.

Recall that to solve for the equilibrium, we must first consider the continuation game after T with no agreement made before and then do backward induction. For the continuation game which is a simultaneous AO bargaining, the limit result is a simultaneous Nash bargaining solution (NBS), which solves (see A.4)

$$\begin{cases} x = \arg\max_s (u(A+s) - u(A))(1-s) \\ x = A \end{cases} \tag{6}$$

Denote the solution of (6) as z^*. Let p_t be the proposal at time t before time T. The backward induction at the limit says p_t satisfies

$$p_T = z^*$$

and

$$ru(p_t) = (2p_t' + r(1-p_t))\Big[(1-e^{-r(T-t)})u'(p_t) + e^{-r(T-t)}u'\big(p_t + \lim_{\Delta \Rightarrow 0} \hat{x}_2(p_t)\big)\big(1 + \lim_{\Delta \Rightarrow 0} \hat{x}_2'(p_t)\big)\Big]$$

where $\lim_{\Delta \Rightarrow 0} \hat{x}_2(p_t)$ is the solution to the Nash bargaining problem $\max_s (u(p_t+s)-u(p_t))(1-s)$. See A.4 for the proof.

Proposition 3 *As the interval between proposals Δ approaches zero and the time before arrival T remains unchanged, the limit result is asymmetric but different from the sequential Nash bargaining solution.*

z^* satisfies $\lim_{\Delta \Rightarrow 0} \hat{x}_2(z^*) = z^*$ because it is the solution to (6). Thus, if z^* is the first price, the outcome will be symmetric, i.e., prices of two bargains are the same. But the backward induction at the limit makes the first price different from z^*, so there is an asymmetric result instead. Moreover, it is not the usual sequential Nash bargaining solution, because this limit result changes with T.

We also look at the limit result of fixing the periods before the arrival of B_2 and letting δ go to 1. Though in our model, two agreements are reached at different dates, when we let the discount factor δ go to 1, the limit result is a simultaneous NBS.

Proposition 4 *As $\delta \to 1$, the equilibrium payoffs of the model go to the symmetric simultaneous NBS – buyers end up paying the flow price z^* which solves (6)*

To see this, we can consider the continuation game where two buyers are active after the date T. As an AO bargaining of one seller bargaining with two buyers simultaneously, its limit result is the NBS of one seller bargaining with two buyers simultaneously. And the symmetric equilibrium result in this continuation game makes the limit outcome symmetric. Furthermore, as $\delta \to 1$, the backward induction before date T does not change the price offered. As a result, the limit result becomes the symmetric simultaneous NBS.

We can also regard the limit of $\delta \to 1$ as making the intervals between proposals approach zero, while fixing the number of intervals before B_2 arrives unchanged. In this explanation, at the limit, all proposals of bargains 1 and 2 tend to be made at one moment. Thus we get simultaneous NBS instead of sequential NBS.

Notice that the sequential NBS is different from the simultaneous NBS. With two Nash bargaining done sequentially, in the second Nash bargaining, the seller's outcome from the first Nash bargaining will be the disagreement point. Let the seller have a payment of x_1 from the first Nash bargaining, then the second Nash bargaining problem is

$$\max_s (u(x_1 + s) - u(x_1))(1 - s)$$

which gives an outcome depending on x_1, denoted as $\hat{x}^N(x_1)$.

Anticipating this, the first Nash bargaining problem is

$$\max_s u(s + \hat{x}^N(s))(1 - s)$$

where the disagreement point of the seller is getting nothing in the bargaining.

When the seller has concave utility, $\hat{x}^N(x_1)$ depends on x_1 and two bargains have different outcomes. For example, with $u(x) = \sqrt{x}$, the outcomes in two bargains are 0.120 and 0.424. The situation is different from the symmetric outcomes of simultaneous NBS.

4 Extension

4.1 Downstream competition

In the main model, buyers do not interact with each other except via bargaining. In this part, we assume that buyers engage in Cournot competition in a downstream market, and bargain with the seller over the price of the input.

4.1.1 Setup

There are three players in the game, one seller and two buyers (B_1, B_2). Buyers buy a critical input from the seller. Assume that 1 unit of input can produce 1 unit of output. The price of that input to each of the two buyers is determined by AO bargaining between each pair. Once concluded successfully, the terms of a bargain are permanent. After completing a bargain with the seller, the first buyer enters the downstream market as a monopolist. B2 arrives at date T (T is odd and $T > 1$). Once B2 arrives and concludes a bargain with the seller, it competes with B1 in the downstream market, assuming Cournot competition. All bargaining uses the same bargaining protocol as in the main model. We still assume the seller has the utility function $u(x)$. The utility functions of buyers are their Cournot profits.

Put in the context of Industrial Organization, the timing of our game allows us to explore the interaction between monopoly pricing and entry. Vertical models such as ours are usually analyzed under one of two different assumptions regarding input pricing. Traditionally, sellers in the input market have been assumed to make take-it-or-leave-it ("TIOLI") offers to buyers of inputs. More recently, models have used the NBS to determine input prices in vertical models. Our contribution is to use AO bargaining in an otherwise standard vertical model. In this spirit, we will sometimes infer to B_1 as the incumbent and to B_2 as the entrant.

Before date T, there is a one-seller-one-buyer AO bargaining between the seller and B_1. After date T, if the seller has already reached an agreement with B_1, then there is simply a one-seller-one-buyer AO bargaining between the seller and B_2. If the seller has not reached an agreement with B_1, then the seller bargains with two buyers simultaneously. In the seller's turn, she proposes two offers to buyers simultaneously. In buyers' turns, the two buyers propose offers to the seller simultaneously.

We assume that the Cournot market has a demand $p = \gamma - \beta(q_1 + q_2)$ each period, where p is the price of the output, q_1, q_2 are the quantities of output by B_1 and B_2 respectively, and γ and β are parameters. When B_1 has a purchasing contract with the seller with an input price c_1 and B_2 does not have a purchasing contract, B_1 is the monopolist in the downstream market. Then B_1 faces with the following problem in each period:

$$\max_q (\gamma - \beta q - c_1) q$$

Clearly, a monopoly buyer with an input price c_1, will choose the quantity of the output $q_m(c_1) = \frac{\gamma - c_1}{2\beta}$ each period, so her payoff in a period $v_m(c_1)$ is

$$v_m(c_1) = \big(\gamma - \beta q_m(c_1) - c_1\big) q_m(c_1) = \frac{(\gamma - c_1)^2}{4\beta} = \beta q_m(c_1)^2,$$

and the seller's utility of this period is $u\big(c_1 q_m(c_1)\big)$.

If both B_1 and B_2 have entered the market with the input prices c_1 and c_2 respectively, then Cournot competition decides their reduced form outputs in one period as $q_1(c_1, c_2)$ and $q_2(c_1, c_2)$:

$$q_1(c_1, c_2) = \frac{\gamma - 2c_1 + c_2}{3\beta}; \; q_2(c_1, c_2) = \frac{\gamma - 2c_2 + c_1}{3\beta}$$

Then buyers' payoffs in this period $v_1(c_1, c_2)$ and $v_2(c_1, c_2)$:

$$v_1(c_1, c_2) = \frac{(\gamma - 2c_1 + c_2)^2}{9\beta} = \beta q_1(c_1, c_2)^2; \; v_2(c_1, c_2) = \frac{(\gamma - 2c_2 + c_1)^2}{9\beta} = \beta q_2(c_1, c_2)^2$$

and the seller's payoff in this period is $u\big(c_1 q_1(c_1, c_2) + c_2 q_2(c_1, c_2)\big)$.

The strategy definition and equilibrium concept are the same as in Section 2.

4.1.2 Equilibrium results

Using the same method as in Section 3.3, we can solve for the equilibrium. Again, Let $\hat{x}^d(x)$ denote the proposal made by the seller to B_2, assuming that the seller has previously negotiated a payment of x with B_1. If the seller has already reached an agreement of flow price x with B_1 before date T, there is a one-seller-one-buyer AO bargaining between the seller and B_2 after date T, then the equilibrium condition for the continuation game after date T is

$$\begin{cases} v_2(x, c_x) = \delta v_2(x, c_y) \\ u\big[x q_1(x, c_x) + c_x q_2(x, c_x)\big] = (1-\delta) u\big[x q_m(x)\big] + \delta u\big[x q_1(x, c_y) + c_y q_2(x, c_y)\big] \end{cases}$$

	$u(x) = 5x - x^2$			$u(x) = \sqrt{x}$		
	Profit of B_1	Output price	Input price	Profit of B_1	Output price	Input price
$T = 5$	0.51591	1.2817	0.56346	0.59309	1.2299	0.45975
$T = 7$	0.51479	1.2825	0.56502	0.60802	1.2202	0.44048
$T = 9$	0.51638	1.2814	0.56281	0.61917	1.2131	0.42626
$T = 11$	0.51922	1.2794	0.55886	0.62758	1.2078	0.41560

Table 1: B_1's profits each period and prices of output before date T

	$u(x) = 5x - x^2$		$u(x) = \sqrt{x}$	
	First input price	Second input price	First input price	Second input price
$T = 5$	0.56346	0.54808	0.45975	0.49134
$T = 7$	0.56502	0.54885	0.44048	0.48149
$T = 9$	0.56281	0.54775	0.42626	0.47420
$T = 11$	0.55886	0.54578	0.41560	0.46874

Table 2: Input prices

where c_x is the input cost proposed by the seller, and c_y is the input cost proposed by B_2. The solution to c_x is the second price $\hat{x}^d(x)$.

In this model, the price in bargain 2 is more likely to increase in the price in bargain 1 compared to the main model. This is because now the input price paid by the incumbent B_1 directly affects the entrant's payoff post entry. If B_1 has a higher production cost, B_2 has less pressure from the competition and is more willing to accept a high input price.

On the other hand, if the seller has not reached an agreement with B_1 at T, then the game becomes a simultaneous bargaining between the seller and two buyers. The seller will propose the same price x^* to two buyers and x^* satisfies $x = \hat{x}^d(x)$.

After knowing the result of the simultaneous bargaining after date T, we can use backward induction to calculate all offers before date T. In each period, the proposer will provide an offer that makes the other player indifferent between accepting and rejecting. In this way we have the seller's proposal x_1^d at date 1, and B_1 will accept that offer immediately. As a result, the seller will provide an offer of $x_2^d = \hat{x}^d(x_1^d)$ at date T, and buyer 2 accepts it immediately.

To see the story behind the algebra above, in Table 1-3 we present the numerical examples with demand $q = 2 - q_1 - q_2$, $\delta = 0.9$, two different utility functions, and four different values

	$u(x) = 5x - x^2$			$u(x) = \sqrt{x}$		
	Profit of B_1	Profit of B_2	Output price	Profit of B_1	Profit of B_2	Output price
$T = 5$	0.22441	0.239219	1.0372	0.27452	0.24242	0.98370
$T = 7$	0.22367	0.239220	1.0380	0.28463	0.24256	0.97399
$T = 9$	0.22472	0.239219	1.0368	0.29221	0.24267	0.96682
$T = 11$	0.22659	0.239216	1.0349	0.29795	0.24276	0.96145

Table 3: Buyers' profits each period and prices of output after date T

of T. Each T corresponds to the length of time from the start of the game until entry occurs. The situation before date T is presented in Table 1, which includes the profit of B_1 each period and the market price of output each period. The input prices each period for two buyers after date T are presented in Table 2. The equilibrium output prices, plus buyers' profits each period after date T are presented in Table 3.

With the IARA utility example ($u(x) = 5x - x^2$), there is a second-mover advantage, but with the DARA utility example ($u(x) = \sqrt{x}$), there is a first-mover advantage. Moreover, with two assumed utility functions in 3 tables, $\hat{x}^d(x)$ is increasing.

We can see that with downstream interaction between buyers, the first price is more likely to boost the second price. In the main model, not like here, the second price is increasing with $u(x) = \sqrt{x}$ and decreasing with $u(x) = 5x - x^2$.

According to Table 3, a larger arrival date T can be in favor of B_1 depending on $u(.)$.

The setup of this game corresponds to a classic problem in industrial organization. Periods before T constitute a pre-entry span of time. Date T commences the post-entry phase. We are concerned with the influence of future entry on the current conduct of a monopolist.

For years, the literature on limit pricing assumes that future entry would discipline pre-entry prices because the incumbent monopolist would wish to convince an entrant that entry is not likely to be profitable.

This informal argument was never entirely convincing, in a world with complete information. With complete information, the entrant can work out for itself what its post-entry profit will be. Why would the pre-entry quantities matter, then? Milgrom and Roberts (1982) formalized a model in which the incumbent has superior information about its own marginal cost. In the Milgrom-Roberts context, the potential entrant has a prior over possible marginal costs of the incumbent. In this setting, the pre-merger price can play a role in signaling the incumbent's costs to the entrant. Without the signaling motive, the possibility of entry would not affect an incumbent's price.

A bargaining setting points to a very different conclusion. See Tables 1-3. In these tables, the first buyer has a downstream monopoly up until period T. At that point, entry occurs with certainty. Each firm knows the costs of the other, and both know the demand curve.

Once entry occurs, if the seller's utility function is the DARA function, the entrant pays a lower input price, the longer T was prior to its entry. Entry causes the first buyer's profit to fall, and the downstream equilibrium output price to fall. Because the first buyer pays less than the second, once entry takes place, its profits are higher than those of the entrant.

These results are quite different from what one would expect if input prices were set according to a TIOLI process. In such a case, the magnitude of T would not affect pricing. Here, it does. Our results also differ from the results in sequential NBS, described by Horn and Wolinsky (1988).

It is also worth noticing that this model could not be set up in a sequential NBS model. Such a model has no place for discounting, or varying the time until entry.

5 Concluding remarks

In this paper, we have extended the Rubenstein AO framework to settings in which different bargaining processes are interdependent. In order to cause one bargain to affect the outcome of another bargain, we have assumed that the seller's utility function is concave in total payments made to the seller.

Propositions 2 describe the equilibrium in which a seller bargains with one buyer before bargaining with another. The equilibrium can exhibit a number of patterns, depending on the seller's utility function.

For example, the second of two sequential bargains to be concluded can lead to either a higher or lower price than the first. Two buyers may pay different prices, even though they are identical. From the standpoint of the seller, sequential bargaining with two buyers can be more or less profitable than bargaining with both simultaneously.

The variety of results stems entirely from assuming that the seller has a nonlinear utility function. If it is linear, then our results coincide with those of Rubenstein.

The usual relationship between AO and NBS exists in sequential bargaining but in a limited way. At each bargaining stage, as the discount factor goes to 1, the bargain at that stage resembles the one that would occur with NBS. However, there is no equivalence across bargaining stages. That is, the limit of the two-stage sequential AO problem is not the same as the sequential NBS.

Finally, we examine the AO version of the vertical model analyzed by Horn and Wolinsky (1988) using the NBS. As they do, we find that identical downstream firms end up paying different prices to an upstream input monopolist. This has implications for the frequent use of profit margins to identify parameters in empirical work. In empirical bargaining models, margins are used to identify Nash bargaining parameters, used in counterfactuals. Our work implies that this approach may lead to biased estimates of these parameters.

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A Appendix

A.1 Bargaining after having an agreement with B_1

Here $\hat{x}_2(x_1)$ is determined by (1).

It is easy to check there is no deviation for players because proposals are making the other players indifferent between accepting and rejecting.

Next, we prove that the equilibrium is unique.

Suppose in an equilibrium, when the seller proposes, the minimal and maximal payoffs are m_s, M_s respectively; when the buyer proposes, the minimal and maximal payoffs are m_b, M_b respectively.

When the seller proposes, the buyer gets at most M_b in the next period, so the proposal by the seller is at least $1 - \delta M_b$. As a result,

$$m_s \geqslant u(x_1 + 1 - \delta M_b) \tag{7}$$

Similarly,

$$M_s \leqslant u(x_1 + 1 - \delta m_b) \tag{8}$$

When the buyer proposes, the seller gets at most $(1-\delta)u(x_1)+\delta M_s$ by rejecting. Because the buyer's proposal is $1 - u_b$, where u_b is the buyer's payoff, we have

$$u(x_1 + 1 - m_b) \leqslant (1 - \delta)u(x_1) + \delta M_s \tag{9}$$

Similarly,

$$u(x_1 + 1 - M_b) \geqslant (1 - \delta)u(x_1) + \delta m_s \tag{10}$$

By (7), (8), (9), and (10) we get

$$u(x_1 + 1 - m_b) \leqslant (1 - \delta)u(x_1) + \delta u(x_1 + 1 - \delta m_b) \tag{11}$$

$$u(x_1 + 1 - M_b) \geqslant (1 - \delta)u(x_1) + \delta u(x_1 + 1 - \delta M_b) \tag{12}$$

Let $f(x) = u(x_1 + 1 - x) - (1 - \delta)u(x_1) - \delta u(x_1 + 1 - \delta x)$.

But

$$\frac{df(x)}{dx} = -u'(x_1 + 1 - x) + \delta^2 u'(x_1 + 1 - \delta x)$$

Because $u'(x) > 0$ and $u''(x) < 0$, $\frac{df(x)}{dx} < 0$. Moreover, $f(0) = u(x_1 + 1) - (1 - \delta)u(x_1) - \delta u(x_1 + 1) > 0$; $f(1) = u(x_1) - (1 - \delta)u(x_1) - \delta u(x_1 + 1 - \delta) < 0$. So, there is a unique solution m^* s.t. $f(x) = 0$ in $[0, 1]$.

Then, (11) and (12) become $m_b \geqslant m^*$ and $M_b \leqslant m^*$, which says $m_b = M_b = m^*$.

Plug $m_b = M_b = m^*$ into (7) and (8) we get $m_s = M_s = u(x_1 + 1 - \delta m^*)$.

So, the equilibrium is unique. By letting $\hat{x}_2(x_1) = 1 - \delta m^*$ and $y = 1 - m^*$, we can see that in this unique equilibrium, the seller's proposal is our $\hat{x}_2(x_1)$, satisfying (1).

A.2 Proof of Proposition 1

By (1), we get

$$u(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) - u(x_1) = \delta(u(x_1 + \hat{x}_2(x_1)) - u(x_1)) \tag{13}$$

Because (13) is satisfied by any $x_1 \in (1 - \delta, 1)$, we can take derivative w.r.t. x_1 on both sides

$$u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})(1 + \frac{\hat{x}_2'(x_1)}{\delta}) - u'(x_1) = \delta(u'(x_1 + \hat{x}_2(x))(1 + \hat{x}_2'(x_1)) - u'(x_1))$$

$$\Leftrightarrow \hat{x}_2'(x_1) = \frac{\delta u'(x_1 + \hat{x}_2(x)) + (1 - \delta)u'(x_1) - u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\frac{u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))}$$

Because $\frac{u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x)) > 0$ by $u''(x) < 0$ and $\delta \in (0, 1)$, we have

$$\hat{x}_2'(x_1) \geqslant 0 \Leftrightarrow \delta u'(x_1 + \hat{x}_2(x)) + (1 - \delta)u'(x_1) - u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) \geqslant 0$$

Moreover,

$$\hat{x}_2'(x_1) > -\frac{u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) - \delta u'(x_1 + \hat{x}_2(x))}{\frac{u'(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta})}{\delta} - \delta u'(x_1 + \hat{x}_2(x))} > -\delta$$

To see the relationship between risk aversion and $\hat{x}_2(.)$, pick $x_1 < x_1'$.

The price $\hat{x}_2(x_1)$ satisfies

$$u(x_1 + 1 - \frac{1 - \hat{x}_2(x_1)}{\delta}) = \delta u(x_1 + \hat{x}_2(x_1)) + (1 - \delta)u(x_1)$$

Firstly, we assume DARA for the seller. Pick $x_1 < x_1'$.

We keep the lottery of getting $\hat{x}_2(x_1)$ with probability δ and getting 0 with probability

$(1-\delta)$ unchanged, and rise the wealth from x_1 to x_1'. Then the expected utility from the lottery is $\delta u(x_1' + \hat{x}_2(x_1)) + (1-\delta)u(x_1')$. According to DARA, the new risk premium π is smaller than the previous risk premium, which is $\frac{1-\delta}{\delta}\Big[1-(1+\delta)\hat{x}_2(x_1)\Big]$. As a result, the new certainty equivalent on the left-hand side is

$$x_1' + \delta\hat{x}_2(x_1) - \pi > x_1' + \delta\hat{x}_2(x_1) - \frac{1-\delta}{\delta}\Big[1-(1+\delta)\hat{x}_2(x_1)\Big] = x_1' + 1 - \frac{1-\hat{x}_2(x_1)}{\delta}$$

Thus, we have The price $\hat{x}_2(x_1)$ satisfies

$$u(x_1' + 1 - \frac{1-\hat{x}_2(x_1)}{\delta}) < \delta u(x_1' + \hat{x}_2(x_1)) + (1-\delta)u(x_1')$$

Notice that $\hat{x}_2(x_1')$ satisfies

$$u(x_1' + 1 - \frac{1-\hat{x}_2(x_1')}{\delta}) = \delta u(x_1' + \hat{x}_2(x_1')) + (1-\delta)u(x_1')$$

So, we have $\hat{x}_2(x_1') > \hat{x}_2(x_1)$, and $\hat{x}_2(.)$ is increasing.

A similar process goes for IARA.

A.3 Simultaneous alternating offer bargaining with two buyers

A sufficient condition for the seller's proposal in equilibrium is

$$1 - x = \delta(1 - (1 - \frac{1-\hat{x}_2(x)}{\delta})) \tag{14}$$

The equation is that the seller's proposal makes the buyer indifferent between accept and reject, where the RHS is the buyer's payoff after rejecting – after rejection, the buyer believes that the other buyer will accept this offer x.

(14) gives $x = \hat{x}_2(x)$. Notice that there must be a solution for this equation in $[1-\delta, 1]$, because $1-\delta \leqslant \hat{x}_2(1-\delta)$ and $1 \geqslant \hat{x}_2(1)$.

Claim 1 *In an equilibrium of this continuation game, the seller will not provide two offers that induce one acceptance and one rejection.*

Proof. Suppose there is an equilibrium such that the seller provides two offers that induce one acceptance and one rejection at date t, denote the price of the acceptance offer as s_1.

The unique equilibrium starting at date $t+1$ is the buyer proposes $1 - \frac{1-\hat{x}_2(s_1)}{\delta}$, and the seller proposes $\hat{x}_2(s_1)$. Clearly, the seller's on-path payoff at date t will be $(1-\delta)(1+\delta)u(s_1) + \delta^2 u(s_1 + \hat{x}_2(s_1))$.

However, if the seller deviates from the rejection offer at date t to the price $\hat{x}_2(s_1)$, the buyer will accept this offer. Thus, this off-path payoff of the seller at date t is $u(s_1 + \hat{x}_2(s_1))$, which is higher than the on-path payoff. As a result, there is a profitable deviation for the seller, which is a contradiction. ■

So, we only need to consider cases where the seller provides offers such that both buyers will accept.

In this case, by a similar reason in the Rubinstein bargaining, the equation (14) characterizes an equilibrium proposal.

The equation is that the seller's proposal makes the buyer indifferent between accept and reject, where the RHS is the buyer's payoff after rejecting – the buyer believes that the other buyer will accept this offer x.

The equation directly gives $x = \hat{x}_2(x)$. Notice that there must be an equilibrium for this equation in $[1-\delta, 1]$, because $1-\delta \leqslant \hat{x}_2(1-\delta)$ and $1 \geqslant \hat{x}_2(1)$.

The above presents a symmetric equilibrium. And the symmetric equilibrium is unique if and only if $\hat{x}_2(x) = x$ has a unique solution.

We now exclude the asymmetric equilibrium, i.e., the seller proposes differently to two buyers.

Suppose the asymmetric equilibrium exists, and the seller proposes x_1 and x_2 to two buyers, where $x_1 \neq x_2$. Then for the same reason as above, they satisfy

$$\begin{cases} x_1 = \hat{x}_2(x_2) \\ x_2 = \hat{x}_2(x_1) \end{cases} \tag{15}$$

Suppose there is an asymmetric equilibrium where the seller proposes x_1 and x_2 to two buyers (w.l.o.g. $x_1 < x_2$), then we must have (x_1, x_2) and (x_2, x_1) on the graph of $\hat{x}_2(x)$.

But as proof in A.2, $f'(x) > -\delta$, we have $\hat{x}_2(x_2) > \hat{x}_2(x_1) - \delta(x_2 - x_1) > x_2 - (x_2 - x_1) = x_1$, contradicting $\hat{x}_2(x_2) = x_1$, so there is no asymmetric equilibrium.

Now we can conclude that the equilibrium determined by (14) is unique when $\hat{x}_2(x)$ has a unique fixed point.

A.4 Limit results

Let first prove the easier case, where $\delta \to 1$ and the number of periods before arrival T remains unchanged, i.e., Proposition 4.

Proof. Consider the continuation game starting at T where the seller does not reach an agreement with B_1. Recall the equilibrium condition for this continuation game (14), for the seller's offer at the limit $\delta \to 1$, it solves

$$x = \lim_{\delta \to 1} \hat{x}_2(x)$$

Notice that $\hat{x}_2(x)$ satisfies (13) at any $\delta \in (0, 1)$, so we can take derivative of both sides of (13) w.r.t. δ:

$$u'(x_1+1-\frac{1-\hat{x}_2(x_1)}{\delta})\frac{\frac{d\hat{x}_2(x_1)}{d\delta}\delta + (1-\hat{x}_2(x_1))}{\delta^2} = u(x_1+\hat{x}_2(x_1))-u(x_1)+\delta u'(x_1+\hat{x}_2(x_1))\frac{d\hat{x}_2(x_1)}{d\delta}$$

Send δ to 1, we get:

$$u'(x_1 + \lim_{\delta \to 1} \hat{x}_2(x_1))(1 - \lim_{\delta \to 1} \hat{x}_2(x_1)) = u(x_1 + \lim_{\delta \to 1} \hat{x}_2(x_1)) - u(x_1) \tag{16}$$

Notice that $\lim_{\delta \to 1} \hat{x}_2(x_1)$ satisfying (16) means:

$$\lim_{\delta \to 1} \hat{x}_2(x_1) = \arg\max_s (u(x_1 + s) - u(x_1))(1 - s)$$

So, the solution to $x = \lim_{\delta \to 1} \hat{x}_2(x)$ solves:

$$\begin{cases} x = \arg\max_s (u(A + s) - u(A))(1 - s) \\ x = A \end{cases}$$

Denote the solution as z^*. The limit outcome of the continuation game starting at T where the seller does not reach an agreement with B_1 is that the seller proposes z^* to both buyers, and buyers accept the offers.

As for the backward induction at date $t < T$, denote the offered prices at date t and $t+1$ are x_t, x_{t+1} respectively. If the seller is the proposer, x_t and x_{t+1} satisfy:

$$1 - x_t = \delta(1 - x_{t+1}) \tag{17}$$

if B_1 is the proposer, x_t and x_{t+1} satisfy:

$$(1-\delta^{T-t})u(x_t)+\delta^{T-t}u(x_t+\hat{x}_2(x_t)) = \delta((1-\delta^{T-t-1})u(x_{t+1})+\delta^{T-t-1}u(x_{t+1}+\hat{x}_2(x_{t+1}))) \tag{18}$$

As δ goes to 1, both (17) and (18) give us $x_t = x_{t+1}$.

Thus, at the limit of $\delta \to 1$, we have that the seller proposes a price of z^* at date 1, and proposes a price of $\lim_{\delta\to 1} \hat{x}_2(z^*) = z^*$ at date T. Both the offers are accepted immediately.

Now we see that the limit payoff of the model is the simultaneous NBS, where both buyers pay z^* to the seller:

$$(1 - \delta^T)u(x_1) + \delta^T u(x_1 + \hat{x}_2(x_1)) \to u(z^* + \lim_{\delta\to 1} \hat{x}_2(z^*)) = u(2z^*),\ as\ \delta \to 1$$

■

Then we look at the limit of the intervals going to zero while the time before arrival remains unchanged. In this process, the number of periods before B_2 arrives goes to infinite.

As for the simultaneous AO bargaining after time T, by the same reason as Proposition 4, is the solution to (6).

We still need to figure out what the backward induction does at the limit.

Let p_t be the proposal made at time t. If p_t is the proposal made by the seller, then it is making the buyer indifferent

$$1 - p_t = e^{-r\Delta}(1 - p_{t+\Delta}) \tag{19}$$

And $p_{t+\Delta}$, the buyer's proposal, is making the seller indifferent

$$\begin{aligned}(1-e^{-r(T-t-\Delta)})u(p_{t+\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+\Delta}+\hat{x}_2(p_{t+\Delta})\big) \\ = (e^{-r\Delta}-e^{-r(T-t-\Delta)})u(p_{t+2\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+2\Delta}+\hat{x}_2(p_{t+2\Delta})\big)\end{aligned} \tag{20}$$

We can rewrite (20) as

$$\begin{aligned}&(1-e^{-r\Delta})u(p_{t+2\Delta}) \\ &= \Big[(1-e^{-r(T-t-\Delta)})u(p_{t+2\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+2\Delta}+\hat{x}_2(p_{t+2\Delta})\big)\Big] \\ &- \Big[(1-e^{-r(T-t-\Delta)})u(p_{t+\Delta}) + e^{-r(T-t-\Delta)}u\big(p_{t+\Delta}+\hat{x}_2(p_{t+\Delta})\big)\Big]\end{aligned} \tag{21}$$

The RHS of (21) is

$$(p_{t+2\Delta}-p_{t+\Delta})\Big[(1-e^{-r(T-t-\Delta)})u'(p_{t+\epsilon}) + e^{-r(T-t-\Delta)}u'\big(p_{t+\epsilon}+\hat{x}_2(p_{t+\epsilon})\big)\big(1+\hat{x}_2'(p_{t+\epsilon})\big)\Big]$$

where $p_{t+\epsilon}$ is between $p_{t+\Delta}$ and $p_{t+2\Delta}$.

Notice that by (19), $p_{t+2\Delta}-p_{t+\Delta} = p_{t+2\Delta}-1+e^{r\Delta}(1-p_t) = p_{t+2\Delta}-p_t+(e^{r\Delta}-1)(1-p_t)$

Thus, (21) is

$$\begin{aligned}&(1-e^{-r\Delta})u(p_{t+2\Delta}) \\ &= \Big[p_{t+2\Delta}-p_t+(e^{r\Delta}-1)(1-p_t)\Big] \\ &\cdot \Big[(1-e^{-r(T-t-\Delta)})u'(p_{t+\epsilon}) + e^{-r(T-t-\Delta)}u'\big(p_{t+\epsilon}+\hat{x}_2(p_{t+\epsilon})\big)\big(1+\hat{x}_2'(p_{t+\epsilon})\big)\Big]\end{aligned} \tag{22}$$

Divide both sides of (22) and send Δ to 0 we get

$$ru(p_t) = (2p_t' + r(1-p_t))\Big[(1-e^{-r(T-t)})u'(p_t) + e^{-r(T-t)}u'\big(p_t+\lim_{\Delta\Rightarrow 0}\hat{x}_2(p_t)\big)\big(1+\lim_{\Delta\Rightarrow 0}\hat{x}_2'(p_t)\big)\Big] \tag{23}$$

This equation and $p_T = z^*$ lead to an asymmetric result after backward induction.

A.5 Proof for $x_1^* < x_U$ with IARA

According to Lemma 1, when the seller has IARA, $\hat{x}_2(x)$ is decreasing.

We know the outcome of the continuation game after T where the seller does not have an agreement with B_1 is the flow price x^*. Denoting the proposed flow price in date t as x_t, we have the following lemma.

Lemma 1 *If the seller has IARA, $x_{T-2} < x_b$.*

Proof. Because T is odd, the backward induction gives us $x_{T-1} = 1 - \frac{1-x_{T-2}}{\delta}$. So, we need to show $1 - \frac{1-x_b}{\delta} > x_{T-1}$ to prove the claim.

Notice that $x^* = \hat{x}_2(x^*)$, $x_b = \hat{x}_2(0)$, and $\hat{x}_2(x)$ decreasing. Thus, we have $x_b > x^*$.

From $x_{T-1} = 1 - \frac{1-x_{T-2}}{\delta}$, then if $1 - \frac{1-x^*}{1-\delta} > x_{T-1}$, we have $x^* > x_{T-2}$. From $x_b > x^*$ we will have the result.

By the backward induction, x_{T-1} satisfies

$$(1-\delta)u(x_{T-1}) + \delta u(x_{T-1} + \hat{x}_2(x_{T-1})) = \delta u(2x^*) \tag{24}$$

It is easy to see the LHS of (24) is increasing in x_{T-1}, so we only need to show

$$(1-\delta)u(1 - \frac{1-x^*}{\delta}) + \delta u(1 - \frac{1-x^*}{\delta} + \hat{x}_2(1 - \frac{1-x^*}{\delta})) > \delta u(2x^*) \tag{25}$$

to prove the lemma.

According to $x^* = \hat{x}_2(x^*)$, x^* satisfies

$$u(x^* + 1 - \frac{1-x^*}{\delta}) = (1-\delta)u(x^*) + \delta u(2x^*)$$

So (25) is equivalent to

$$(1-\delta)[u(1 - \frac{1-x^*}{\delta}) + u(x^*)] + \delta u(1 - \frac{1-x^*}{\delta} + \hat{x}_2(1 - \frac{1-x^*}{\delta})) > u(x^* + 1 - \frac{1-x^*}{\delta}) \tag{26}$$

Because $\hat{x}_2(x)$ is decreasing, $\hat{x}_2(1 - \frac{1-x^*}{\delta}) > \hat{x}_2(x^*) = x^* \Rightarrow u(1 - \frac{1-x^*}{\delta} + \hat{x}_2(1 - \frac{1-x^*}{\delta})) > u(1 - \frac{1-x^*}{\delta} + x^*)$. This proves $x^* > x_{T-2}$.

Moreover, by the concavity of $u(x)$, $u(1-\frac{1-x^*}{\delta})+u(x^*)>u(1-\frac{1-x^*}{\delta}+x^*)$

Because $x_b>x^*$, (26) holds and $x_b>x_{T-2}$. ■

Lemma 2 *For any odd number $\tau<T$:*

$$(1-\delta^\tau)u(1-\frac{1-x_b}{\delta})+\delta^\tau u(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))\geqslant(\delta-\delta^\tau)u(x_b)+\delta^\tau u(x_b+\hat{x}_2(x_b)) \quad (27)$$

Proof. According to the equilibrium condition of the unaffected bargaining, (27) is equivalent to

$$(1-\delta^\tau)\delta u(x_b)+\delta^\tau u(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))\geqslant(\delta-\delta^\tau)u(x_b)+\delta^\tau u(x_b+\hat{x}_2(x_b))$$

$$\Leftrightarrow \delta u(x_b)(\delta^{\tau-1}-\delta^\tau)+\delta^\tau u(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))\geqslant\delta^\tau u(x_b+\hat{x}_2(x_b))$$

$$\Leftrightarrow (1-\delta)u(x_b)+u(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))\geqslant u(x_b+\hat{x}_2(x_b))$$

$$\Leftrightarrow u(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))-u(1-\frac{1-x_b}{\delta})\geqslant u(x_b+\hat{x}_2(x_b))-u(x_b)$$

$$\Leftrightarrow u(x_b)-u(1-\frac{1-x_b}{\delta})\geqslant u(x_b+\hat{x}_2(x_b))-u(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta})) \quad (28)$$

By the concavity of $u(x)$,

$$RHS\ of\ (28)<[\frac{1-\delta}{\delta}(1-x_b)+\hat{x}_2(x_b)-\hat{x}_2(1-\frac{1-x_b}{\delta})]u'(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))$$

$$LHS\ of\ (28)>\frac{1-\delta}{\delta}(1-x_b)u'(x_b)$$

But $\frac{1-\delta}{\delta}(1-x_b)<1-\delta<p_2(1-\frac{1-x_b}{\delta})\Rightarrow x_b<1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta})$.

$$\Rightarrow u'(x_b)>u'(1-\frac{1-x_b}{\delta}+\hat{x}_2(1-\frac{1-x_b}{\delta}))>0$$

Moreover, because $\hat{x}_2(x)$ is decreasing,

$$\frac{1-\delta}{\delta}(1-x_b)+\hat{x}_2(x_b)-\hat{x}_2(1-\frac{1-x_b}{\delta})<\frac{1-\delta}{\delta}(1-x_b)$$

Thus, (28) holds. ■

Lemma 3 *If the seller has IARA, for some odd number $t < T$, if $x_t < x_b$, then $x_{t-2} < x_b$.*

Proof. Let us look at an $x_t < x_b$ where t is odd and $t < T$. The backward induction implies

$$(1 - \delta^{T-t+1})u(x_{t-1}) + \delta^{T-t+1}u(x_{t-1} + \hat{x}_2(x_{t-1})) = (\delta - \delta^{T-t+1})u(x_t) + \delta^{T-t+1}u(x_t + p_2 * (x_t))$$

Due to $x_t < x_b$ and Lemma 2, we have

$$\begin{aligned}
(\delta - \delta^{T-t+1})u(x_t) + \delta^{T-t+1}u(x_t + \hat{x}_2(x_t)) \\
&< (\delta - \delta^{T-t+1})u(x_b) + \delta^{T-t+1}u(x_b + \hat{x}_2(x_b)) \\
&\leqslant (1 - \delta^{T-t+1})u(1 - \frac{1 - x_b}{\delta}) + \delta^{T-t+1}u(1 - \frac{1 - x_b}{\delta} + \hat{x}_2(1 - \frac{1 - x_b}{\delta}))
\end{aligned}$$

Thus, $x_{t-1} < 1 - \frac{1-x_b}{\delta}$. But $x_{t-2} = 1 - \delta(1 - x_{t-1})$ by backward induction, so $x_{t-2} < x_b$. ■

According to Lemma 1 and Lemma 3, if the seller is not prudent, for any odd number $t < T$, we have $x_t < x_b$, which means $x_1 < x_b$.

Moreover, because $\hat{x}_2(x)$ is decreasing with a non-prudent seller, $\hat{x}_2(x_1) < \hat{x}_2(0) = x_b$. As a result, both flow prices are smaller than the flow price in the AO benchmark, and thus the seller is worse off than having two separate AO bargaining.