Cheap Talk and Advertising with Naive Receivers

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Abstract

Based on the cheap talk model with naive receivers who take the message at face value in Ottaviani and Squintani (2006), I endogenize the probability of the receiver blindly believing in the sender by allowing the sender to increase this naivety probability at a cost. When the probability chosen is observed by receivers, receivers can benefit from this ability of the sender, and the fully revealing equilibrium is possible. But this ability of the sender damages information transmission and removes the fully revealing equilibrium if the probability is not observable. These results can explain how information is conveyed in advertising when the advertiser can design the content of advertising as well as use extra expenditure to affect the consumers' gullibility.

Keywords: Information transmission, Cheap talk, Naive receivers, Deception

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1 Introduction

Crawford and Sobel (1982) introduced the classical cheap talk model, where both players are fully rational. Ottaviani and Squintani (2006) introduced a variant, where the receiver may be naive, i.e. the receiver takes the message at face value. In this paper, building on Ottaviani and Squintani (2006), I allow for the possibility that the sender can increase the naivety probability at a cost, i.e. convince the receiver of his credibility at a cost.

This model has one sender (he) and receivers (she) of mass 1, but there are two possible types of receivers, rational and naive. Informed of the state of the world, the sender chooses the message to send and the probability of receivers being naive. A cost will be incurred if the sender chooses a positive naivety probability. Finally, a rational receiver takes action strategically based on the message while a naive receiver takes the message at face value. The payoffs of both sides are determined by the state and the actions of the receivers.

In Ottaviani and Squintani (2006), they discussed a cheap talk with a possibly naive receiver, but their naivety probability is given exogenously. Our innovation is making the naivety probability endogenous. Now the sender can increase the naivety probability at a cost. Compared with their exogenous model, now the behavior of the receiver is not only affected by the belief induced by the sender's message but also directly affected by the sender's strategy on another dimension—the naivety probability. It captures the fact that the sender might try to convince the receiver of his credibility and now the content of messages in the model matters as discussed in Sobel (2013).

This model can provide some new insights into advertising. We usually discuss two effects of advertising. One of them is saying that advertising can alter consumers' tastes towards products and can have anti-competitive consequences (Bagwell, 2007, p. 1711, p. 1720). This effect is discussed by Braithwaite (1928), Comanor and Wilson (1974) and Stigler and Becker (1977), as well as some more recent experiments like Elder and Krishna (2010) and McClure et al. (2004). Another effect is saying that advertising can provide information and can have a pro-competitive effect (Bagwell, 2007, p. 1716). Huh, DeLorme and Reid (2004), King et al. (1987) and Taylor (2011) had discussed the informativeness of advertising. But this model looks at another aspect—the effect of advertising on consumers' gullibility—to provide some theoretical insights. People are aware of these deceiving attempts by advertising, which can limit the deceiving power of advertising (Cain, 2011; Friestad and Wright, 1994; Petty and Andrews, 2008; Rotfeld, 2008). However, there are techniques that advertisers may use to reduce the defense of consumers and thus make them more naive such as embedded advertising (Cain, 2011). And Johar (1995) looked at under which conditions consumers can be deceived.

Also, there is an important theory that regards advertising as a dissipative signal studied by Milgrom and Roberts (1986) and Nelson (1974). As discussed in Anand and Shachar (2009), the money-burning view on advertising has two main critiques, one is that the expenditure on advertising has no effect other than the role of signal in this perspective. And another one is that the content of advertising does not matter. In response to these two critiques, Anand and Shachar (2009) added targeting and noisiness to the informative advertising model. By allowing the advertiser to choose the message and a non-dissipative expenditure, my model is also robust to the two critiques mentioned above. The content of the message matters due to the existence of naive receivers, and the expenditure has a role other than a signal. Moreover, not like in Anand and Shachar (2009), this model also allows the advertiser to design the content of advertising.

The probability of naive receivers chosen by the sender can be observed by the receivers since consumers are aware of the deception effort by the sender (Friestad and Wright, 1994) and can infer the effect of the sender's deception by the effort. Consumers can see the effort exerted by the sender from the content and the design of the advertisement. This can happen particularly if the target consumers of the advertisement have professional knowledge in communication, such as journalists or other advertisers. When the probability is observable, the equilibrium result is similar to Ottaviani and Squintani (2006). With unbounded state space, fully separating equilibria are possible. With bounded state space, we can have partitional equilibria as in Crawford and Sobel (1982) and hybrid equilibria where only states below a threshold are fully revealed. Compared to the standard cheap talk model on which my model nests, the ability of the sender to increase the naivety probability promotes information transmission and increases the welfare of receivers.

However, it is also possible that the probability chosen is not observed by receivers. For example, the ability to defend deception can be weakened by techniques like embedded advertising (Cain, 2011). Also, when consumers are not sure about where the advertisement has been placed, consumers are not able to infer how many consumers may be cheated since they do not know the general characteristics of the audience. When the probability is not observable, a fully separating section cannot exist in equilibrium. Now the sender's deviation in the probability will not affect the rational receivers' actions. Though Ottaviani and Squintani (2006) had shown that a higher probability of the naive receiver increases the equilibrium welfare of both sides, here the sender has the incentive to deviate to probability 0 (when the on-path probability is positive), since the existence of naive receivers also serves as a cost of lying in messages. And without naive receivers, the situation is similar to Crawford and Sobel (1982) and thus no fully separating section can be supported. Though classic partitional equilibria can still be supported in this case, making naivety probability unobservable has weakened information transmission and reduced the welfare of receivers, compared to the observable case.

These results suggest that advertising with deceptive power may not always hurt consumers. As long as consumers are aware of the advertiser's effort to deceive them, this deceiving ability of the sender can promote information transmission. Intuitively, when the probability is observable, the sender can effectively commit to a cost of lying in messages—there exist naive receivers who take the message at face value and whose actions would be too high for the sender when the sender further exaggerates the message. On the contrary, if the probability is not observable, the sender cannot choose a positive probability in equilibria—the sender would like to make the rational receivers believe that there exist naive receivers who are the cost of lying, but removing naive receivers benefits the sender.

This paper is also following the literature discussing non-strategic players in cheap talk, which can make the model a signaling game. Crawford (2003) visited the case with boundedly rational players to study the lying for strategic advantage. Kartik, Ottaviani and Squintani (2007) explored the cheap talk with the naive receiver who is credulous and Kartik (2009) looked into the situation where lying is costly for the sender. These two set-ups grant the fully separating equilibrium when the support of the state's distribution is unbounded above. And Ottaviani and Squintani (2006) found a new category of equilibrium—the hybrid equilibrium—for the cheap talk with a possibly naive receiver in the bounded state support, capturing some features of the fully separating equilibrium. And their discussions also inspire equilibria selection called NITS in Chen, Kartik and Sobel (2008). Chen (2011) added the exogenously honest sender as well as the naive receiver to cheap talk and discuss the strategic communication.

Austen-Smith and Banks (2000) and Kartik (2007) combined the cheap talk and money-burning signal and assume the sender can 'burn' some money as a signal. But the naivety probability chosen by the sender in my model is different from their signal, which is a purely dissipative one. Besides reducing the payoff as a cost and altering the payoff via the belief of the receiver, the naivety probability also directly changes the payoff of the sender by changing the composition of the receiver types.

The organization of this paper is as follows. Section 2 provides the basic setup of the model. Section 3 and Section 4 explore the existence of equilibria when the naivety probability is observed and not observed by receivers respectively. Section 5 introduces some extensions of the model. Finally, Section 6 is the concluding remarks.

2 Model

I consider a cheap talk game with an informed sender (he) and uninformed receivers (she) of mass 1, where their payoffs depend on the actions of receivers and the state. There exist two types of receivers, naive and rational. A naive receiver takes the message she receives at face value. A rational receiver is Bayesian rational. And the sender can increase the naivety probability at a cost.

At the beginning of the game, the state s is drawn from $\mathcal{S} \subset \mathbb{R}$ according to a given distribution. Then the sender observes the state s privately and sends a message $m \in \mathcal{M} = \mathbb{R}$ to the receivers. And he also chooses the naivety probability $p \in \mathcal{E}$, which incurs a cost $c(p)$. Receiving the message sent by the sender, naive receivers and rational receivers choose actions $a_n, a_r \in \mathcal{A} = \mathbb{R}$, which finally decide the payoffs of both sides. The strategy of the sender is the pair $\gamma = (\sigma : \mathcal{S} \to \Delta(\mathcal{M}),\ \varepsilon : \mathcal{S} \to \Delta(\mathcal{E}))$. I will assume that rational receivers observe the naivety probability chosen by the sender when making decisions in Section 3 and then look at the case with unobservable naivety probabilities in Section 4. For the observable case, the strategy of rational receivers is $\alpha : \mathcal{M} \times \mathcal{E} \to \Delta(\mathcal{A})$. In the unobservable case, the strategy of rational receivers is $\alpha : \mathcal{M} \to \Delta(\mathcal{A})$.

I make some simplifying assumptions to have a more clear insight. When $\mathcal{S}$ is bounded, I assume s to be uniformly distributed on $\mathcal{S} = [0, U]$. As for unbounded $\mathcal{S}$, I assume it to be $\mathcal{S} = [0, \infty)$. Also, the message m needs to be chosen from $\mathcal{S}$. And players' payoffs are quadratic-loss and not aligned:

- the sender's payoff is

$$u(a_r, a_n, p, s) = -p(a_n - s - b)^2 - (1 - p)(a_r - s - b)^2 - c(p)$$

- receivers' payoff is

$$v(a, s) = -(a - s)^2$$

Here b measures the conflict of interest between the sender and receivers. The sender's optimal action is $s+b$ while receivers' optimal action is s.

It is easy to see that receivers' optimal action would be their expectation of the state. For naive receivers, they take the message m at face value, so their action and belief are both equal to m. But for rational receivers, they will form a strategical belief about the state and thus choose the action $E(s|m)$.

The cost function $c(.) : \mathcal{E} \to [0, \infty)$ has following properties: (*i*) it is twice continuously differentiable; (*ii*) it is strictly increasing and strictly convex, i.e. $c'(p) > 0$, $c''(p) > 0$; (*iii*) $\mathcal{E} = [0, \bar{p})$ and $\bar{p} < 1$; (*iv*) $c(0) = 0$ and $c(p), c'(p) \to \infty$ as $p \to \bar{p}$. The first two properties are standard. Property (*iii*) and (*iv*) are saying that the model nests the standard cheap talk model of Crawford and Sobel (1982)—if the sender invests no cost, there is no naive receiver. Also, the sender is not able to fully cheat receivers—there is an upper bound smaller than 1 for the naivety probability and the cost goes to infinity as the probability approaches the upper bound.

Formally, a perfect Bayesian equilibrium (γ, α, μ) consists of the sender's strategy on the message and the probability $\gamma(s) = (\sigma(.|s),\ \varepsilon(.|s))$, the rational receiver's action $\alpha(m)$, and belief $\mu(.|m)$[1] (or $\mu(.|p, m)$ in the observable case) such that:

- for each s, $\gamma(s) = (\sigma(.|s),\ \varepsilon(.|s))$ solves $\max_{\sigma,\varepsilon} \mathbf{E}_{\sigma,\varepsilon} u(\alpha(m), m, p, s)$,
- for each m, $\alpha(m)$ solves $\max_a \int_{\mathcal{S}} v(a, s)\mu(s|m)ds$,
- $\mu(.|m)$ is Bayesian whenever possible.

Also, the action of a receiver is always a pure strategy since $\frac{\partial^2 v}{\partial a^2} < 0$. I slightly abuse the notation and make $\sigma(s) = m$ ($\varepsilon(s) = p$) mean that $\sigma(.|s)$ ($\varepsilon(.|s)$) puts probability 1 on m (p). And I note $\mu(m)$ as the mean of the distribution $\mu(.|m)$.

[1] $\sigma(.|s)$, $\varepsilon(.|s)$ and $\mu(.|m)$ are PDF for continuous distributions and PMF for discrete distributions

3 Observable Probability

I consider the version of the model where the effort and thus the naivety probability p is observed by receivers first. This assumption is corresponding to the case where the listener knows the effort behind the communication attempt, which is more likely to happen when the listener herself has some experience in communicating such messages.

The equilibrium results of this case are similar to Ottaviani and Squintani (2006). We will have fully separating equilibria when the state support is unbounded above and will have hybrid and partitional equilibria when the state support is bounded. In a partitional equilibrium, the state space will be partitioned into several parts and the message sent by the sender will reveal which elements of the partition the state falls in. In a hybrid equilibrium, the states below a threshold will be fully revealed and the states above the threshold will be partitioned into finitely many sections. The formal definitions of these three kinds of equilibria are:

Definition 1 (i) *In the* Fully Separating Equilibrium*, the messages sent fully reveal the states*

(ii) *In the* Hybrid Equilibrium*, there is a threshold s^* in the state support such that there is full separation when $s < s^*$ and there is a finite partition when $s \geqslant s^*$*

(iii) *In the* Partitional Equilibrium*, there is a finite partition of the state support, and the message reveals which element of the partition the state belongs to.*

The following proposition summarizes the equilibria with observable naivety probabilities.

Proposition 1 *When the probability is observed by the receivers:*

(i) When $\mathcal{S} = [0, \infty)$, if $\exists\ p^* > 0$ such that $\frac{b^2}{p^{*2}} = c'(p^*)$ and $\frac{1-p^*}{p^*} b^2 + c(p^*) \leqslant b^2$, then there exists a fully separating equilibrium.

(ii) When $\mathcal{S} = [0, U]$, if $U > 0$, $\exists\ \underline{b}_1 > 0$, such that $\forall b < \underline{b}_1$, there is a hybrid equilibrium.

(iii) When $\mathcal{S} = [0, U]$, $\forall b > 0$, there is a partitional equilibrium.

Here $\underline{b}_1$ is related to the specification of $c(.)$ and U, and a detailed proof can be found in Appendix A1. Though in the analysis by Ottaviani and Squintani (2006), a higher probability of naive receivers makes players better off, in the construction of equilibria in my model, we need to punish the behavior of choosing $p = 0$ and thus put some restrictions on the existence of equilibria. This is because we have to prevent the sender from removing naive receivers, who are also the cost of lying.

The construction of the hybrid equilibrium is largely depending on the interest conflict b. As b goes to 0, the number of sections in the partition above the threshold goes to infinity and the lengths of sections in the partition go to 0. This is consistent with normal intuition. When the interest conflict between the receiver and the sender is pretty small, the information conveyed can be relatively precise. Also, the fully separating messages are less inflated with smaller b.

Similar to Ottaviani and Squintani (2006), one problem worth attention to is that the off-path belief is not fully convincing. I have set the belief of observing small off-path messages to be high, and the belief of observing large off-path messages to be small.

For any partitional equilibrium in Crawford and Sobel (1982), there is a corresponding partitional equilibrium that generates the same result here. Note that since such an equilibrium has the same outcome as Crawford and Sobel (1982), their restriction on the number of sections in the partition also applies here. Only small interest conflicts can support a large number of parts in the partition. The off-path belief problem also exists in the partitional equilibrium: a small off-path message will lead to a large belief.

When the probability p is observable, the sender can effectively commit to a cost of lying—naive receivers whose actions will be too high for the sender if the sender further exaggerates the message. As a result, we can see that in this model a full separation section is possible in equilibrium, but in the standard cheap talk the model nests on, there is no fully separating section in equilibria. As long as rational receivers can observe the naivety probability chosen by the sender, the sender's deceptive power can promote information transmission and increase the welfare of rational receivers. In the application of advertising, this result suggests that deceptive attempts by the advertiser can help information transmission if consumers are aware of the advertiser's effort on affecting them.

4 Unobservable Probability

The naivety probability can also be unobservable, since consumers may have little image of how advertising affects people. Or they are not sure where the advertisement has been placed and thus have no idea about the general characteristics of the audience, which are essential to estimate how many people will be affected by the advertisement.

When the probability p is not observed by receivers, the fully separating equilibrium does not exist even when $\mathcal{S} = [0, \infty)$. And the hybrid equilibrium does not exist for $\mathcal{S} = [0, U]$ if the interest conflict b is small. But we can still find partitional equilibria in bounded $\mathcal{S}$.

Proposition 2 *When the probability is not observed by receivers:*
(i) When $\mathcal{S} = [0, \infty)$, the fully separating equilibrium cannot exist for any $b > 0$.
(ii) When $\mathcal{S} = [0, U]$, $\exists\ \underline{b}_2 > 0$, $\forall b < \underline{b}_2$, the hybrid equilibrium does not exist.
(iii) When $\mathcal{S} = [0, U]$, $\exists\ \underline{b}_3 > 0$, $\forall b < \underline{b}_3$, the partitional equilibrium exists if the number of parts in the partition is large enough.

Here $\underline{b}_2$ and $\underline{b}_3$ are related to the specification of $c(.)$ and U. The proof for the proposition is in Appendix A2.

When the probability is not observed by receivers, we are not able to punish the choice of $p = 0$, so full separation will be a difficulty in an equilibrium. Now choosing $p = 0$ secretly will not affect the action of rational receivers, so the sender will choose to remove naive receivers and thus the cost of lying for the sake of his benefit. Then without the existence of naive receivers, a fully separating equilibrium does not exist.

If receivers can commit to taking the message at face value like in Ottaviani and Squintani (2006), a better information transmission can be achieved and players are better off. The sender's ability to choose $p = 0$ not observed without extra cost is an inability to commit to a cost of lying. Lack of commitment power removes the fully separating equilibrium and damages information transmission, thus reducing welfare.

Moreover, even the babbling equilibrium cannot exist for arbitrary parameter specifications if $\mathcal{S} = [0, \infty)$. Here the sender seeing states large enough can have a profitable deviation from the babbling strategy and thus break the equilibrium. When $\mathcal{S} = [0, U]$, if the sender can reach a naivety probability very close to 1 with very small effort, the babbling equilibrium will break down as well (see Appendix A3 for detailed proof).

So, when consumers are not aware of the naivety probability chosen by the advertiser, less information can be conveyed in advertising, and rational consumers' welfare decreases compared to observable cases. And the advertiser here does not want to use his deceptive power—he would choose naivety probability 0.

5 Expansion

In this section, I will discuss the extension in which the sender will choose the naivety probability before knowing the state. The probability chosen beforehand is corresponding to the long-run cost. In the example of a salesman, accumulating human resources like speech skills should be considered as a long-run cost compared to the time the salesman spends with the consumer for selling, since it is usually done before deciding the details of the selling attempt like which product to sell, which group to aim at, etc.

5.1 Observable Probability Chosen beforehand

In this part, I will let the sender choose the naivety probability before knowing the state. Now the timing of the game is: firstly, the sender chooses the probability of naive receivers p at a cost $c(p)$. Then the state s is decided by nature and is observed by the sender. Observing the state, the sender chooses the message m sent to the receivers. And finally, receivers take actions (a_r and a_n respectively by rational and naive receivers) and payoffs are realized. This set-up is representing the case of long-run investment, like a salesman investing in his human capital, or an advertising company hiring an experienced design team. As a long-run cost, here I focus on pure probability strategies.

Since the continuation subgame after the sender chooses the probability p is the same as in Ottaviani and Squintani (2006), the equilibrium of the subgame can be partitional or hybrid. In a partitional equilibrium, there will be a partition of $\mathcal{S}$ and the message reveals which part of the partition the state falls in. In a hybrid equilibrium, the messages will fully reveal the states below a threshold and reveal the states above the threshold in the same way as in a partitional equilibrium. As a result, if naivety probability is observable, players can use a babbling equilibrium in the continuation subgame as a punishment device to support a large set of equilibria for the whole game.

Proposition 3 *When $\mathcal{S} = [0, U]$ and p is observed by receivers,*
(i) $\exists\ \underline{b}_4 > 0$ such that $\forall b < \underline{b}_4$, there exists a hybrid equilibrium.
(ii) $\exists\ \underline{b}_5 > 0$ such that $\forall b < \underline{b}_5$, there exists a partitional equilibrium.

We refine the equilibria of the continuation subgame according to the highest expected payoff of rational receivers since they are the receivers making strategic decisions. Once the sender chooses the probability p, the continuation game can have one hybrid equilibrium and multiple partitional equilibria. By the criterion of the highest expected payoff of rational receivers, among the partitional equilibria, the one with the largest number of parts survives, so we just need to compare this one with the hybrid equilibrium. It turns out that rational receivers always prefer the hybrid equilibrium. Note that fixing b, only some p can support a hybrid equilibrium in the continuation subgame[2]. As for those p too small to support a hybrid equilibrium, the refinement result would be the partitional equilibrium with the largest number of elements.

[2]See Ottaviani and Squintani (2006) part 4.1.4

Applying such refinement to the continuation subgame, we will have a unique equilibrium for the model except for some knife-edge cases.

Proposition 4 *If $\mathcal{S} = [0, U]$ and p is observed by receivers, and only the subgame equilibrium that is optimal for rational receivers is selected in the continuation subgame, there will be a unique equilibrium, which is either hybrid or partitional.*

We can also use the sender's payoff as a criterion to refine the equilibria in the continuation game. If we only select the subgame equilibrium of the continuation subgame that is optimal to the sender, we will have a similar unique equilibrium as in Proposition 4. However, the favorite subgame equilibrium of the sender may be different from the one preferred by rational receivers.

5.2 Unobservable Probability Chosen beforehand

Once the naivety probability becomes unobservable to receivers, receivers cannot punish the deviation in the probability. This is a lack of commitment power and again restricts the scope of equilibria.

Proposition 5 *When $\mathcal{S} = [0, U]$ and p is not observed by receivers,*
(i) $\exists\ \underline{b}_6 > 0$, $\forall b < \underline{b}_6$, the hybrid equilibrium does not exist;
(ii) $\exists\ \underline{b}_7 > 0$, $\forall b < \underline{b}_7$, the partitional equilibrium exists if the number of parts in the partition is large enough.

This is very similar to Proposition 2. Once the probability is not observable, whether it is chosen before or after knowing the state does not affect its role in information transmission a lot. Long-run effort and short-run effort that the advertiser exerts to deceive consumers are both hindering the information transmission if they are not observed by consumers.

6 Conclusion

In this paper, I build a model of cheap talk with endogenous naivety probability of receivers. In this cheap talk, some of the receivers are naive, i.e. taking the message at face value. And the sender can increase the naivety probability at a cost. Out of simplification, I assume that the state space is bounded with a uniform distribution or unbounded above, and the utility is quadratic loss. Under these assumptions, I explore the existence of equilibria.

If the naivety probability chosen by the sender is observed by receivers, then we can have a fully separating equilibrium if the state space is unbounded. And in the bounded state space, we can have a hybrid equilibrium in which there is a fully separating section. A fully separating section is possible here because the sender can commit to a lying cost with observable naivety probability. Compared to the standard cheap talk on which the model nests, we can see that the sender's deceptive power has made fully separating possible at least for some of the states. So, the information transmission is promoted and it benefits rational receivers. On the contrary, if the probability chosen is not observed by receivers, there is no commitment power and such benefit does not exist. We can lose fully separating equilibria and hybrid equilibria. Also, in the case of unobservable probability, the sender tends to choose $p = 0$, and the equilibria are similar to the ones in the standard cheap talk.

If we make the sender choose the naivety probability before knowing the state instead of after that, we can have a large set of equilibria with observable probability—many naivety probabilities can be supported in equilibria—because we can use the babbling equilibrium in the continuation subgame after choosing the probability as a useful punishment device. But if we focus on sender-optimal or rational-receiver-optimal equilibria in the continuation subgame, a unique equilibrium can be chosen. If the probability is not observable, then the equilibrium results are similar to the case with naivety probability chosen after knowing the state.

If we apply this model to advertising, the results above are suggesting that the attempt of the advertiser to deceive consumers (making them less sophisticated) may not always hurt the information transmission. If consumers are aware of the advertiser's attempt, the attempt can grant better information transmission compared to the case where the advertiser does not have this ability. However, if the probability chosen by the advertiser is not observed by consumers, consumers will be more defensive and the advertiser will not use his ability, and the equilibrium information transmission will be quite similar to the case without this deceptive power.

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A Mathematical Appendix

A.1 Proposition 1

(i) The construction of a fully separating equilibrium is:

The sender chooses the message $\sigma(s) = s + \frac{b}{p^*}$ and the probability $\varepsilon(s) = p^*$, where $\frac{b^2}{p^{*2}} = c'(p^*)$.

The rational receiver has the belief $\mu(m,p) = \begin{cases} m - \dfrac{b}{p}, & m \geqslant \dfrac{b}{p}, p > 0 \\ \dfrac{b}{p} + L(p), & m < \dfrac{b}{p}, p > 0 \end{cases}$ and $\mu(m,p) = 0, p = 0$.

The rational receiver's action is $a(m,p) = \mu(m,p)$

Here $L(p)$ is large enough such that $p(1-p)L(e)^2 \geqslant \frac{1-p^*}{p^*}b^2 + c(p^*)$

Fixing the strategy of the sender, the rational receiver's strategy is optimal.

Fixing the strategy of the rational receiver, according to equation (1), observing any s, the sender's payoff of choosing m, p such that $m \geqslant \frac{b}{p}$ and $p > 0$ is:

$$U(m,p;s,b,\gamma) = -p(m-s-b)^2 - (1-p)(m - \frac{b}{p} - s - b)^2 - c(p)$$

So we have $\frac{\partial U(m,p;s,b,\gamma)}{\partial m} = -2(m - s - \frac{b}{p}) = 0 \Rightarrow m^*(p) = s + \frac{b}{p} \geqslant \frac{b}{p} \Rightarrow U(m^*(p),p;s,b,\gamma) = -\frac{1-p}{p}b^2 - c(p)$ [3]. And $\frac{\partial U(m^*(p),p;s,b,\gamma)}{\partial p} = \frac{1}{p^2}b^2 - c'(p) = 0 \Rightarrow b^2 = p^2c'(p)$. So, $m(s) = s + \frac{b}{p^*}$, $p = c(p^*)$ is the best choice among $m \geqslant \frac{b}{p}$, $p > 0$ for any s.

With off-path strategies of $0 \leqslant m < \frac{b}{p}$ and $p > 0$, for any s, the deviation payoff is $-p(m-s-b)^2 - (1-p)(\frac{b}{p} + L(p) - s - b)^2 - e \leqslant -p(1-p)(\frac{b}{p} + L(p) - m)^2 < -p(1-p)L(e)^2 \leqslant -\frac{1-p^*}{p^*}b^2 - c(p^*)$, so they are not profitable.

The off-path strategies with $p = 0$ are stopped by $-\frac{1-p^*}{p^*}b^2 - c(p^*) \geqslant -b^2$.■

[3] $U(m,p;s,b,\gamma) \leqslant -\frac{1-p}{p}b^2 - c(p)$ here.

(ii) To see the existence of the hybrid equilibrium, we need this lemma:

Lemma 1 *If* $U > \sqrt{c'(0)}$*, for* ε *such that* $U - \sqrt{c'(0)} > \varepsilon > 0$*,* $\exists\ b_1(\varepsilon) > 0$*, such that* $\forall 0 < b < b_1(\varepsilon)$*,* $\exists\ p^* > 0$ *such that* $p^{*2}c'(p^*) = b^2$ *and* $U > \frac{b}{p^*} + \varepsilon$

Proof. We have $c''(p) > 0$ and $U > \sqrt{c'(0)}$, then by the continuity of $\sqrt{c'(p)}$, for a certain ε satisfying $U - \sqrt{c'(0)} > \varepsilon > 0$, $\exists\ \underline{p} > 0$ such that $U > \sqrt{c'(\underline{p})} + \varepsilon$, let $b_1(\varepsilon)^2 = \underline{p}^2 c'(\underline{p})$.

We also have $c''(p) > 0$, $c'(p) > 0$, $c(0) = 0$ and $c'(0) > 0$, so $p^2c'(p)$ is increasing in p and $p^2c'(p) = 0$ when $p = 0$. Then by the continuity of $p^2c'(p)$, $\forall 0 < b < b_1(\varepsilon)$, $\exists\ 0 < p^* < \underline{p}$ such that $p^{*2}c'(p^*) = b^2$. Also, $\frac{b}{p^*} = \sqrt{c'(p^*)} < \sqrt{c'(\underline{p})} < U - \varepsilon$. ■

Lemma 1 ensures that the construction below is possible.

The specific construction of a hybrid equilibrium is:

The sender chooses the message

$$\sigma(s) = \begin{cases} s + \dfrac{b}{p^*}, & s \in [0, a_0) \\ m_i, & s \in [a_{i-1}, a_i)\ (i = 1, 2, ...N) \end{cases}$$

and the probability

$$\varepsilon(s) = \begin{cases} p^*, & s \in [0, a_0) \\ 0, & s \in [a_0, U] \end{cases}$$

The rational receiver's on-path action is

$$\alpha(m, p) = \begin{cases} m - \dfrac{b}{p}, & m \in [\dfrac{b}{p^*}, a_0 + \dfrac{b}{p^*}), p = p^* \\ \dfrac{a_{i-1} + a_i}{2}, & m = m_i, p = 0 \end{cases}$$

and her off-path belief is $\mu(m, p) = \begin{cases} \dfrac{a_{i-1} + a_i}{2}, & m = m_i, p > 0 \\ \dfrac{U + a_{N-1}}{2}, & m \in [\dfrac{b}{p^*}, a_0 + \dfrac{b}{p^*})\ \&\ p \neq p^* \end{cases}$ and $\mu(m, p) =$

$$\begin{cases} \frac{U+a_{N-1}}{2}, & m < \frac{b}{p^*} \\ \frac{a_0+a_1}{2}, & m \geqslant a_0 + \frac{b}{p^*} \ \& \ m \neq m_i \end{cases}$$

Here $m_i \neq m_j$, $i \neq j$ and $m_i \approx U$, $p^{*2}c'(p^*) = b^2$, and approximately

$a_i = a_0 + (U - a_0)\frac{i}{N} - 2i(N-i)b$, $a_0 = U - 2N^2 b - 2N\sqrt{\frac{1-p^*}{p^*}b^2 + c(p^*)}$,

$$N \in \left(\frac{1}{2}\sqrt{\frac{3-p^*}{p^*} + \frac{c(p^*)}{b^2} + \frac{2\varepsilon}{b}} - \frac{1}{2}\sqrt{\frac{1-p^*}{p^*} + \frac{c(p^*)}{b^2}}, \right.$$
$$\left. \frac{1}{2}\sqrt{\frac{1-p^*}{p^*} + \frac{c(p^*)}{b^2} + \frac{2U}{b}} - \frac{1}{2}\sqrt{\frac{1-p^*}{p^*} + \frac{c(p^*)}{b^2}}\right)$$ [4]

To verify that it is indeed a equilibrium, firstly note that fixing the strategy of the sender, the rational receiver does not deviate.

Fixing the strategy of the rational receiver, by the argument similar to part (i), the sender will not turn to the deviations with on-path messages and on-path naivety probabilities of states in the fully separating part.

Furthermore, for deviations with on-path messages within the partitional pooling part, we need the sender at cutoff points to be indifferent between the strategies of the upper area and the lower area. To be specific, the indifference condition needs to be satisfied at $a_0, a_1, ..., a_{N-1}$.

The indifference condition at a_i, $i = 1, 2, ..., N-1$ is (A1):

$$\max_p -p(m_i - a_i - b)^2 - (1-p)\left(\frac{a_{i-1} + a_i}{2} - a_i - b\right)^2 - c(p) =$$
$$\max_p -p(m_{i+1} - a_i - b)^2 - (1-p)\left(\frac{a_{i+1} + a_i}{2} - a_i - b\right)^2 - c(p)$$

The indifference condition at a_0 is (A2):

$$-\frac{1-p^*}{p^*}b^2 - c(p^*) = \max_p -p(m_1 - a_0 - b)^2 - (1-p)\left(\frac{a_1 + a_0}{2} - a_0 - b\right)^2 - c(p)$$

Note that for $N \geqslant 2$ we need (A1) and (A2) and for $N = 1$, we only need (A2).

[4] ε is the one in Lemma 1. In one equilibrium, I fix one ε such that $0 < \varepsilon < U - \frac{1}{\sqrt{P'(0)}}$.

Firstly, consider the case $N \geqslant 2$. Take the derivative of LHS of (A1) w.r.t. p we can get (A3): $(a_{i-1} + a_i - 2U)(\frac{a_{i-1}+a_i+2U}{4} - a_i - b) - c'(p)$.

If $b < \frac{a_{i-1}-3a_i+2U}{4}$, we have $(A3) < 0$ and thus optimal effort level $e = 0$ for LHS. $b < \frac{a_{i-1}-3a_i+2U}{4}$ also leads to $b < \frac{a_{i+1}-3a_i+2U}{4}$, so $e = 0$ is optimal for RHS as well.

With $a_i = a_0 + (U - a_0)\frac{i}{N} - 2i(N-i)b$, we have

$$b < \frac{a_{i-1} - 3a_i + 2U}{4} \Leftrightarrow (U - a_0)\frac{2N - 2i - 1}{N} > b(1 + (2i+1)(2i+1-2N))$$

We have $i = 1, 2, ..., N-1$ and $N \geqslant 2$, so $2N - 2i - 1 > 0$ and $1 + (2i+1)(2i+1-2N) \leqslant \max\{1 + 3(3-2N), 1 - (2N-1)\} < 0$. As a result, $b < \frac{a_{i-1}-3a_i+2U}{4}$ is satisfied for $i = 1, 2, ..., N-1$. So, (A1) is equivalent to $(\frac{a_{i-1}+a_i}{2} - a_i - b)^2 = (\frac{a_{i+1}+a_i}{2} - a_i - b)^2$. And we have

$$\begin{aligned} a_i = a_0 + (U - a_0)\frac{i}{N} - 2i(N-i)b &\Rightarrow a_{i+1} - a_i = a_i - a_{i-1} + 4b \\ &\Rightarrow (\frac{a_{i-1} + a_i}{2} - a_i - b)^2 = (\frac{a_{i+1} + a_i}{2} - a_i - b)^2 \end{aligned}$$

So the indifference condition at a_i ($i = 1, 2, ..., N-1$) is satisfied in the construction.

As for the indifference condition at a_0, we have $b < \frac{a_0-3a_1+2U}{4}$, so $b < \frac{a_1-3a_0+2U}{4}$. (A2) is now equivalent to $-\frac{1-p^*}{p^*}b^2 - c(p^*) = -(\frac{a_1+a_0}{2} - a_0 - b)^2$. Note that we also need $-\frac{1-p^*}{p^*}b^2 - c(p^*) > -(\frac{a_1+a_0}{2} - x - b)^2$ when x is slightly below a_0, so $\frac{a_1+a_0}{2} - a_0 - b > 0$ is needed, which means that now (A2) requires $a_0 = U - 2N^2b - 2N\sqrt{\frac{1-p^*}{p^*}b^2 + c(p^*)}$. And this condition is satisfied in the construction.

For the case of $N = 1$, (A2) becomes $-\frac{1-p^*}{p^*}b^2 - c(p^*) = \max_p -p(U - a_0 - b)^2 - (1-p)(\frac{U+a_0}{2} - a_0 - b)^2 - c(p)$. It is easy to verify that a_0 in the construction makes the optimal effort of RHS equal to 0. And such a_0 also satisfies $-\frac{1-p^*}{p^*}b^2 - c(p^*) = -(\frac{U+a_0}{2} - a_0 - b)^2$ and the requirement that states slightly below a_0 prefer fully separating messages, so the indifference condition at a_0 is satisfied.

To make the construction reasonable, we also need $a_0 > 0$ and $a_0 + \frac{b}{p^*} < U - \varepsilon$ (the ε in Lemma 1). Plugging in the specification of a_0 we will have the condition $N \in$

$(\frac{1}{2}\sqrt{\frac{3-p^*}{p^*}+\frac{c(p^*)}{b^2}+\frac{2\varepsilon}{b}}-\frac{1}{2}\sqrt{\frac{1-p^*}{p^*}+\frac{c(p^*)}{b^2}}, \frac{1}{2}\sqrt{\frac{1-p^*}{p^*}+\frac{c(p^*)}{b^2}+\frac{2U}{b}}-\frac{1}{2}\sqrt{\frac{1-p^*}{p^*}+\frac{c(p^*)}{b^2}})$, which is the restriction for N. With $U > \frac{b}{p^*}+\varepsilon$ (can be achieved with b small enough according to Lemma 1), this interval is not empty and its length goes to infinity as $b \to 0$.

And it is a quick result that when b is small enough, $e = 0$ is optimal for any $s \geqslant a_0$ with the on-path message. This result, together with the analysis above, ensures that the deviations with on-path messages within the partitional pooling part are not profitable for any s.

Then I will deal with the remaining off-path deviations. Note that the on-path payoff is either $-\frac{1-p^*}{p^*}b^2 - c(p^*)$ or $-(\frac{a_{i-1}+a_i}{2} - s - b)^2$. Since as $b \to 0$, we have $a_i - a_{i-1} \to 0$ and $c(p^*) \to 0$, the on-path payoff goes to 0 as $b \to 0$.

For the off-path deviation choosing $m \in [\frac{b}{p^*}, a_0 + \frac{b}{p^*})$,

$$\begin{aligned}
U^D(s) &\leqslant \max_p -p(m-s-b)^2 - (1-p)(\frac{U+a_{N-1}}{2} - s - b)^2 \\
&= \max\{-(\frac{U+a_{N-1}}{2} - s - b)^2, -\bar{p}(m-s-b)^2 - (1-\bar{p})(\frac{U+a_{N-1}}{2} - s - b)^2\}
\end{aligned}$$

Note that $-(\frac{U+a_{N-1}}{2} - s - b)^2$ is always no larger than the on-path payoff, and as $b \to 0$:

$$\begin{aligned}
-\bar{p}(m-s-b)^2 - (1-\bar{p})(\frac{U+a_{N-1}}{2} - s - b)^2 &\to -\bar{p}(m-s)^2 - (1-\bar{p})(U-s)^2 \\
&\leqslant -\bar{p}(1-\bar{p})(U-m)^2 < -\bar{p}(1-\bar{p})\varepsilon^2 < 0 \ (m < a_0 + \frac{1}{\sqrt{P'(0)}} < U - \varepsilon)
\end{aligned}$$

so when b is small enough, for any $m \in [\frac{b}{p^*}, a_0 + \frac{b}{p^*})$, $p \neq p^*$ and s, the deviation payoff will not be profitable.

When deviating to $0 \leqslant m < \frac{b}{p^*}$, the off-path payoff is

$$\begin{aligned}
U^D(s) &\leqslant \max_p -p(m-s-b)^2 - (1-p)(\frac{U+a_{N-1}}{2} - s - b)^2 \\
&= \max\{-(\frac{U+a_{N-1}}{2} - s - b)^2, -\bar{p}(m-s-b)^2 - (1-\bar{p})(\frac{U+a_{N-1}}{2} - s - b)^2\}
\end{aligned}$$

As $b \to 0$, we have:

$$-\bar{p}(m-s-b)^2-(1-\bar{p})(\frac{U+a_{N-1}}{2}-s-b)^2 \to -\bar{p}(m-s)^2-(1-\bar{p})(U-s)^2$$
$$\leqslant -\bar{p}(1-\bar{p})(U-m)^2 < -\bar{p}(1-\bar{p})\varepsilon^2 < 0 \; (m < \frac{b}{p^*} \to \frac{1}{\sqrt{P'(0)}} < U-\varepsilon).$$

So when b small enough, for any $0 \leqslant m < \frac{b}{p^*}$ and s, $U^D(s)$ is no larger than the on-path payoff (note that $-(\frac{U+a_{N-1}}{2}-s-b)^2$ is always no larger than the on-path payoff). So, no sender will deviate to $m < \frac{b}{p^*}$.

When deviating to $m \in [a_0+\frac{b}{p^*}, U]$ & $m \neq m_i$, the off-path payoff

$$U^D(s) \leqslant \max\{-(\frac{a_0+a_1}{2}-s-b)^2, -\bar{p}(m-s-b)^2-(1-\bar{p})(\frac{a_0+a_1}{2}-s-b)^2\}$$

As $b \to 0$:

$$-\bar{p}(m-s-b)^2-(1-\bar{p})(\frac{a_0+a_1}{2}-s-b)^2 \to -\bar{p}(m-s)^2-(1-\bar{p})(a_0-s)^2$$
$$\leqslant -\bar{p}(1-\bar{p})(m-a_0)^2 \leqslant -\bar{p}(1-\bar{p})c'(0) < 0$$

As a result, when b is small enough, for any $m \in [a_0+\frac{b}{p^*}, U]$ & $m \neq m_i$ and s, the deviation is not profitable (note that $-(\frac{a_0+a_1}{2}-s-b)^2$ is always no larger than the on-path payoff). So, no sender will deviate to $m \geqslant a_0+\frac{b}{p^*}$ & $m \neq m_i$.

So when b is small enough, for any state s, the deviation payoff is no larger than the on-path payoff. ■

(iii) The construction of a partitional equilibrium is:

The sender chooses the message m uniformly from $[a_{i-1}, a_i)$ if $s \in [a_{i-1}, a_i)$, and the probability is always $p = 0$.

The rational receiver's on-path action is $\alpha(m;\gamma) = \frac{a_{i-1}+a_i}{2}$ for $m \in [a_{i-1}, a_i)$ and $p = 0$. The off-path belief is $\mu(m,p) = \frac{\frac{a_{i-1}+a_i}{2}-pm}{1-p}$ for $m \in [a_{i-1}, a_i)$ and $p > 0$.

Here $a_i = \frac{iU}{N} - 2i(N-i)b$ and $U > 2N(N-1)b$.

Fixing the strategy of the sender, the rational receiver's strategy is optimal.

Fixing the strategy of the rational receiver, directly applying the result of Crawford and Sobel (1982), we will have that the sender has no incentive to do on-path deviations. If he (with any s) deviates to the off-path strategy of $m \in [a_{i-1}, a_i)$ and $p > 0$, the payoff will be $U^D(s) = -p(m-s-b)^2 - (1-p)(\frac{\frac{a_{i-1}+a_i}{2} - pm}{1-p} - s - b)^2 - c(p) \leqslant -(\frac{a_{i-1}+a_i}{2} - s - b)^2 - c(p)$ (concavity). And this works for any i, so for any s, the off-path deviations with $m \in \mathcal{S}$ and $p > 0$ will not be profitable.■

A.2 Proposition 2

(i) Suppose there is a fully separating equilibrium with the sender's strategy $\sigma(.|s)$ and $\varepsilon(.|s)$. Since it is fully separating, $\text{supp}(\sigma(.|s_i)) \cap \text{supp}(\sigma(.|s_j)) = \varnothing$ if $s_i \neq s_j$. So, the optimal response of the rational receiver is $\alpha(m;\gamma) = s$ for $m \in \text{supp}(\sigma(.|s))$, while the response of the naive receiver is m. And these must be their strategies in this equilibrium (since they are optimal fixing the sender's strategy).

So the sender's on-path payoff with s is strictly negative since $-p(m-s-b)^2 - (1-p)(\alpha(m;\gamma) - s - b)^2 - c(p) = -p(m-s-b)^2 - (1-p)b^2 - c(p) < 0$, $\forall m \in \text{supp}(\sigma(.|s))$ and $\forall p \in \text{supp}(\varepsilon(.|s))$. However, if he deviates to $m' \in \text{supp}(\sigma(.|s+b))$ and $p = 0$ (pretending to be the sender observing the state $s+b$), the rational action is $s+b$ and the naive action is m'. So, this deviation payoff is $-0 \times (m'-s-b)^2 - 1 \times (s+b-s-b)^2 - 0 = 0$, and thus is profitable. We also have that $\text{supp}(\sigma(.|s+b))$ is not empty since $s+b$ is always in the state support unbounded above. As a result, this deviation is feasible. Now we have that in any candidate fully separating equilibrium, for the sender of any state s, there is a profitable deviation. ■

(ii) Consider a candidate hybrid equilibrium and note the fully separating part as $[0, a_0)$.

For $s < a_0$, we have $-p(m-s-b)^2-(1-p)(\alpha(m;\gamma)-s-b)^2-c(p) = -p(m-s-b)^2-(1-p)b^2-c(p) \leqslant -(1-p)b^2-c(p)$ ($\forall m \in \text{supp}(\sigma(.|s))$, $\forall p \in \text{supp}(\varepsilon(.|s))$). Taking derivative we can get $b^2 - c'(p)$. Since $c'(p)$ is bounded by $c'(0)$, when b is small enough, $b^2 - c'(p) < 0$ for any p. In this case, the on-path payoff will be no larger than $-b^2$ for $s < a_0$. For the sender with the state $s < a_0$, when deviating to $p = 0$ and $m' \in \text{supp}(m(.|s+\varepsilon))$ (here $s+\varepsilon < a_0$ and $b > \varepsilon > 0$), the deviation payoff is $-0 \times (m'-s-b)^2 - 1 \times (s+\varepsilon-s-b)^2 = -(b-\varepsilon)^2 > -b^2$, so it is a profitable deviation. So, the equilibrium cannot be held here when b is small. ■

(iii) In the partitional equilibrium, the state support is divided into N parts and cutoff points are noted as $0 = a_0 < a_1 < ... < a_N = U$, where $a_i = \frac{iU}{N} - 2i(N-i)b$ and $U > 2N(N-1)b$.

The sender's strategy $\sigma(m|s)$ is a uniform distribution among $[a_{i-1}, a_i)$ if $s \in [a_{i-1}, a_i)$[5], and $\varepsilon(s) = 0$ for any s.

The rational receiver's belief is $\mu(m) = \frac{a_{i-1}+a_i}{2}$ for $m \in [a_{i-1}, a_i)$ and the action is $\alpha(m) = \mu(m)$, while the naive receiver chooses action m.

Fixing the strategy of the sender, the rational receiver will not deviate.

Fixing the rational receiver's strategy, the sender will not deviate to other messages in $\mathcal{S}$ if p stays at 0, since the partition is following Crawford and Sobel (1982). If he deviates to $p > 0$ and $m \in \mathcal{S}$, his deviation payoff with $s \in [a_{i-1}, a_i)$ is

$$\begin{aligned} U^D(s) &\leqslant \max_m -p(m-s-b)^2 - (1-p)(\frac{a_{j-1}+a_j}{2} - s - b)^2 - c(p) \\ &\leqslant -(1-p)(\frac{a_{i-1}+a_i}{2} - s - b)^2 - c(p) := \bar{U}^D(s) \end{aligned}$$

[5]When $i = N$, this becomes $[a_{N-1}, U]$.

We have $a_i = \frac{iU}{N} - 2i(N-i)b$ and $U > 2N(N-1)b$, as a result:

$$\begin{aligned}\frac{\partial \bar{U}^D(s)}{\partial p} &= (\frac{a_{i-1}+a_i}{2} - s - b)^2 - c'(p) \leqslant (\frac{a_{i-1}-a_i}{2} - b)^2 - c'(p) \\ &= (\frac{U}{2N} - (N-2i)b)^2 - c'(p) \leqslant (\frac{U}{2N} + Nb)^2 - c'(p) \leqslant (\frac{U}{N} + b)^2 - c'(p)\end{aligned}$$

$c'(p)$ is bounded by $c'(0)$, so $\frac{\partial \bar{U}^D(s)}{\partial p}$ will be negative for any p and $s \in [a_{i-1}, a_i)$ with b small enough and N large enough[6]. So, the deviation payoff of $s \in [a_{i-1}, a_i)$ will be no larger than the on-path payoff, since the derivative is negative for any p. And this works for any i[7], so no state wants to deviate to $p > 0$, $m \in \mathcal{S}$ when b is small enough and N is large enough. So, there is no profitable deviation with $m \in \mathcal{S}$ for the sender. ■

A.3 Babbling Equilibrium with Unobservable Probability

(i) When the probability is not observable, we can conclude that the babbling equilibrium does not exist for the state support unbounded above.

To prove this, first note that if $Es = \infty$, the rational receiver has no optimal reaction.

If $Es < \infty$, firstly let us look at the pure message strategy, assume m to be the on-path message

Then we have the on-path payoff:

$$U(s) = \max_p -p(m-s-b)^2 - (1-p)(Es-s-b)^2 - c(p)$$

Pick an arbitrary probability level $p_1 \in (0, \bar{p})$ and off-path message $m' = s + b$ (s is large enough such that $m' \neq m$), then the off-path payoff is (with the harshest punishment: $a_r = 0$)[8]

[6]Note that the restriction for N is $U > 2N(N-1)b$, so N can be arbitrarily large when b is arbitrarily small.

[7]Once N and b let $(\frac{U}{N} + b)^2 - c'(0) < 0$, $\frac{\partial \bar{U}^D(s)}{\partial p} < 0$ for any s and p, no matter which part i the state s is in.

[8]If some punishments are $a_r > b$ and $a_r \neq Es$, there must be some states that the sender wants to deviate to the off-path message and $p = 0$, so the punishment for the off-path messages should be smaller than b. As a result, the harshest punishment is 0. As for the punishment $a_r = Es$, the argument for $a_r = 0$

$$V(s) = -(1-p_1)(-s-b)^2 - c(p_1)$$

Then we have $|U(s)| \geqslant p(s)(m-s-b)^2 + (1-p(s))(Es-s-b)^2$ ($p(s) = \arg\max_p U(s)$) and $|V(s)| \leqslant (1-p_1)(s+b)^2 + c(p_1)$, so:

$$\lim_{s\to\infty} \frac{|U(s)|}{|V(s)|} \geqslant \frac{p(s)(m-s-b)^2 + (1-p(s))(Es-s-b)^2}{(1-p_1)(s+b)^2 + c(p_1)} \to \frac{1}{1-p_1} > 1$$

So for very large states there will be a profitable deviation and there is no pure strategy babbling equilibrium

Then if the message strategy is mixed

Note that if the strategy mixes on m_1 and m_2, then the optimal probability corresponding to these two messages has to be 0:

w.l.o.g. $m_1 < m_2$

Let $F_1^s(p) = -p(m_1-s-b)^2 - (1-p)(Es-s-b)^2 - c(p)$ and $F_2^s(p) = -p(m_2-s-b)^2 - (1-p)(Es-s-b)^2 - c(p)$

Since we mix on m_1 and m_2, we have $\max_p F_1^s(p) = \max_p F_2^s(p)$ for any s

For $s+b > \frac{m_1+m_2}{2}$: $F_1^s(p) < F_2^s(p)$ for $p > 0$

$\Rightarrow p_1^* = 0$ (otherwise $\max_p F_1^s(p) = F_1^s(p_1^*) < F_2^s(p_1^*) \leqslant \max_p F_2^s(p)$)

$\Rightarrow \max_p F_2^s(p) = F_1^s(0) = F_2^s(0)$

Then, $\frac{F_2^s(p)}{dp}|_{p=0} \leqslant 0 \Rightarrow p_2^* = 0$

Similarly, for $s+b < \frac{m_1+m_2}{2}$, we also have $p_1^* = p_2^* = 0$

And by the continuity of $\frac{dF_i^s(p)}{dp}$, $s+b = \frac{m_1+m_2}{2}$ will also have $p_1^* = p_2^* = 0$

Then we can see that the mixed message cannot be larger than Es, since with $m_1 > Es$, $\lim_{s\to\infty}[(Es-s-b)^2 - (m_1-s-b)^2] - c'(0) = \infty > 0$. Then the optimal probability cannot be 0.

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still works.

But by the same argument as the pure strategy case, if all on-path message is no larger than Es, large states have profitable deviations

So there is no mixed strategy babbling equilibrium either.

(ii) When the probability is not observable, if the specification of $c(.)$ satisfies that $c'(0)$ and $c''(p)$ are small enough and $\bar{p}$ is close to 1 enough, then the babbling equilibrium cannot exist for the bounded state support with $b < U$.

To see this, we need to find $c'(0)$ and $c''(p)$ small enough and $\bar{p}$ close enough to 1 to make all possible message strategies not babbling equilibrium

(1) Pure message strategy

(i) On-path message $m = \frac{U}{2}$

If the punishment $a' \geqslant b$, the for $s = a' - b$ can have 0 off-path payoff. So the on-path payoff for $s = a' - b$ should also be 0, then we have $a' = \frac{U}{2}$.

Since we need off-path payoff no larger than on-path payoff, then for any s and m':

$$
\begin{aligned}
&\max_p -p(m' - s - b)^2 - (1-p)(\frac{U}{2} - s - b)^2 - c(p) \leqslant -(\frac{U}{2} - s - b)^2 \\
&\Rightarrow [(\frac{U}{2} - s - b)^2 - (m' - s - b)^2] - c'(0) \leqslant 0
\end{aligned}
$$

It is easy to see that small $c'(0)$ can make this not satisfied for $s = U$ and $m = U$

If the punishment $a' < b$, we just need to consider the harshest punishment $a' = 0$

Then for $s \in [0, U - b]$, we need $\max_p -(1-p)(s+b)^2 - c(p) \leqslant -(\frac{U}{2} - s - b)^2$ (off-path payoff $\leqslant$ on-path payoff)

But making $c'(0)$ and $c''(p)$ small enough and $\bar{p}$ close enough to 1, this inequality can be violated for some s.

(ii) On-path message $m \neq \frac{U}{2}$

If the punishment $a' \geqslant b$, then similarly we have $a' = \frac{U}{2}$

Then for $s \in [0, U-b]$ but $s + b \neq m$, the on-path payoff is $U(s) = \max_p -p(m-s-b)^2 - (1-p)(\frac{U}{2} - s - b)^2 - c(p)$, but its off-path payoff can be $V(s) = \max_p -(1-p)(\frac{U}{2} - s - b)^2 - c(p)$, we need to have $V(s) \leqslant U(s)$, so the on-path optimal effort has to be 0 and thus $U(s) =$

$-(\frac{U}{2} - s - b)^2$

Then we need $V(s) \leqslant U(s) = -(\frac{U}{2}-s-b)^2$ and it is trivial that $V(s) \geqslant -(\frac{U}{2}-s-b)^2$. Now we know that $-(1-p)(\frac{U}{2}-s-b)^2-c(p)$ is maximized at $p=0$ and thus $(\frac{U}{2}-s-b)^2-c'(0) \leqslant 0$

But we can find $c'(0)$ small enough to make this necessary condition not satisfied.

If the punishment $a' < b$, we just need to consider the worst punishment $a' = 0$

For $s \in [0, U-b]$, the off-path payoff can be $V(s) = \max_p -(1-p)(s+b)^2 - c(p)$ and the on-path payoff is $U(s) = \max_p -p(m-s-b)^2 - (1-p)(\frac{U}{2}-s-b)^2 - c(p) \leqslant \max\{-(m-s-b)^2, -(\frac{U}{2}-s-b)^2\}$.

This means that we can find $s \in [0, U-b]$ such that on-path payoff $U(s) \leqslant -(\frac{U-b}{4})^2$, but by making $c'(0)$ and $c''(p)$ small enough and $\bar{p}$ close enough to 1, we can ensure $V(s) > -(\frac{U-b}{4})^2$ for any $s \in [0, U-b]$

(2) Fully mixed message

As argued in section (i), the optimal effort corresponding to each on-path message has to be 0

So $p = 0$ has to maximized $-p(m-s-b)^2 - (1-p)(\frac{U}{2}-s-b)^2 - c(p)$ for $m = U$ and $s = U$, which means that we need $[(\frac{U}{2}+b)^2 - b^2] - c'(0) \leqslant 0$

But when $c'(0)$ is small enough, this is not satisfied

(3)Mixed message

If the punishment $a' \geqslant b$, then by the same argument as before $a' = \frac{U}{2}$

Again, we have that any on-path message is corresponding to a zero optimal effort, so the on-path payoff is $U(s) = -(\frac{U}{2} - s - b)^2$

Then we have that the off-path payoff $V(s) = \max_p -p(m'-s-b)^2 - (1-p)(\frac{U}{2}-s-b)^2 - c(p) \leqslant -(\frac{U}{2}-s-b)^2$

As a result, for these off-path messages, the optimal efforts are still 0

Now for any s and m, the optimal effort needs to be 0, which is $[(\frac{U}{2}-s-b)^2 - (m-s-b)^2] - c'(0) \leqslant 0$ for any s and m

But with $c'(0)$ small enough, we can make this not satisfied for $s = U$ and $m = U$

If the punishment $a' < b$, we just need to consider the worst punishment $a' = 0$

Similarly, we can let $c'(0)$ and $c''(p)$ small enough and $\bar{p}$ close enough to 1 to make $\max_p -(1-p)(s+b)^2 - c(p) > -(\frac{U}{2} - s - b)^2$ for $s + b = U$ and $m = U$, then $m = U$ cannot be an off-path message

And we can have $c'(0)$ small enough such that $[(\frac{U}{2} - s - b)^2 - (m - s - b)^2] - c'(0) > 0$ for $m = s = U$, then $m = U$ cannot be an on-path message.

Now we can conclude that for $c'(0)$ and $c''(p)$ small enough and $\bar{p}$ close to 1 enough, no message strategy can be a babbling equilibrium in the bounded state space.

A.4 Proposition 3

(i) The construction of the equilibrium is:

The sender chooses probability p^* and the message

$$\sigma(s) = \begin{cases} s + \dfrac{b}{p^*}, & s \in [0, a_0) \\ m_i, & s \in [a_{i-1}, a_i) \ (i = 1, 2, ...N) \end{cases}$$

The rational receiver's on-path action is

$$\alpha(m, p) = \begin{cases} m - \dfrac{b}{p}, & m \in [\dfrac{b}{p^*}, a_0 + \dfrac{b}{p^*}), p = p^* \\ \dfrac{a_{i-1} + a_i}{2}, & m = m_i, p = p^* \end{cases}$$

and her off-path belief is $\mu(m, p) = \frac{\frac{U}{2} - pm}{1-p}, p \neq p^*$

and $\mu(m, p) = \begin{cases} U, & m < \dfrac{b}{p^*}, p = p^* \\ 0, & m \geqslant a_0 + \dfrac{b}{p^*} \ \& \ m \neq m_i, p = p^* \end{cases}$

Here $m_i \neq m_j$ for $i \neq j$ and $m_i \approx U$, p^* satisfies $\frac{b^2}{p^{*2}} = c'(p^*)$ and $c(p^*) < \mathbf{E}[(\frac{U}{2} - s)^2]$, and approximately

$a_i = a_0 + (U - a_0)\frac{i}{N} - 2i(N-i)b$, $a_0 = U - b\left(\frac{1+p^*+\sqrt{(1+p^*)^2-4(p^*+\frac{1-p^*}{4N^2})(p^*+(N^2-\frac{1}{p^*})(1-p^*))}}{2(p^*+\frac{1-p^*}{4N^2})})\right)$,
$N = \max\{n \in \mathbb{N} : p^* < \frac{1}{2n(n-1)}\}$

According to Proposition 1 in Ottaviani and Squintani (2006), players have no profitable deviation in the subgame after the sender choosing the naivety probability p when b is small enough, so we just need to ensure that the sender has no incentive to deviate from the on-path probability p^*.

If the sender deviates to $p \neq p^*$, then his best payoff in the continuation game is $\mathbf{E}[-(\frac{U}{2} - s - b)^2]$, as b goes to 0, his payoff goes to $\mathbf{E}[-(\frac{U}{2} - s)^2] - c(p)$.

But the sender's on-path payoff in the continuation game is $-\frac{1-p^*}{p^*}b^2$ for $s < a_0$ and $-(1-p^*)(\frac{a_{i-1}+a_i}{2} - s - b)^2 - p^*(U - s - b)^2$ for $s \in [a_{i-1}, a_i)$, and $a_0 \to U$ as $b \to 0$. So the sender's on-path payoff goes to $-c(p^*)$ as b goes to 0.

We have $-c(p^*) > \mathbf{E}[-(\frac{U}{2} - s)^2] \geqslant \mathbf{E}[-(\frac{U}{2} - s)^2] - c(p)$, so the sender has no incentive to deviate in p when b is small enough.

(ii) The construction of the equilibrium is:

The sender chooses the probability p^* and the message $\sigma(s) = \frac{a_{i-1}+a_i}{2} := m_i$

The rational receiver's on-path action is $\alpha(m, p) = m,\ m = m_i$ & $p = p^*$,and her off-path belief is $\mu(m, p) = \frac{\frac{U}{2} - pm}{1-p},\ m \neq m_i\ or\ p \neq p^*$

Here $a_i = \frac{iU}{N} - 2i(N-i)b,\ i = 0, 1, 2, ..., N,\ N = \max\{n \in \mathbb{N} : U > 2n(n-1)b\}$

According to the proof for the Proposition 2 in Ottaviani and Squintani (2006), players have no profitable deviation in the subgame after the sender choosing the naivety probability p when b is small enough. Again, we just need to ensure that the sender has no incentive to deviate from the on-path probability p^*.

As b goes to 0, the number of elements in the partition N goes to infinity, which makes the payoff from the continuation game after choosing p go to 0. As a result, the on-path payoff of the sender goes to $-c(p^*)$ as b goes to 0.

But the best continuation payoff when the sender deviates to $p \neq p^*$ is $\mathbf{E}[-(\frac{U}{2} - s - b)^2]$, so the limit of the sender's off-path payoff is $\mathbf{E}[-(\frac{U}{2} - s)^2] - c(p) \leqslant \mathbf{E}[-(\frac{U}{2} - s)^2] < -c(p^*)$. So, the sender will not deviate in p when b is small enough.

A.5 Proposition 4

In the continuation game after the sender choosing p, there can be a hybrid equilibrium and multiple partitional equilibria as discussed in Ottaviani and Squintani (2006).

In a hybrid subgame equilibrium, the rational receiver's payoff is:

$$U_h = \sum_{i=1}^{N_h} \int_{a_{i-1}}^{a_i} -(\frac{a_{i-1} + a_i}{2} - s)^2 ds = -(\frac{(U - a_0)^3}{12N_h^2} + \frac{1}{3}b^2(U - a_0)(N_h + 1)(N_h - 1)))$$

In a partitional subgame equilibrium, the rational receiver's payoff is:

$$U_p = \sum_{i=1}^{N_p} \int_{a_{i-1}}^{a_i} -(\frac{a_{i-1} + a_i}{2} - s)^2 ds = -(\frac{U^3}{12N_p^2} + \frac{1}{3}b^2 U(N_p + 1)(N_p - 1)))$$

According to Ottaviani and Squintani (2006) section 4.1.4, for each b, $\exists\ P(b)$ such that only $p \in P(b)$ can support a hybrid equilibrium. And such $p(b) \in P(b)$ must satisfy $a_0 = U - b\left(\frac{1+p(b)+\sqrt{(1+p(b))^2-4(p(b)+\frac{1-p(b)}{4N_h^2})(p(b)+(N_h^2-\frac{1}{p(b)})(1-p(b)))}}{2(p(b)+\frac{1-p(b)}{4N_h^2})})\right) > 0$.

As a result, when p chosen cannot support a hybrid subgame equilibrium, if we use rational receiver's payoff as refinement criterion on subgame equilibria, we will have a partitional equilibrium with $N = \max\{n \in \mathbb{N} : U > 2n(n-1)b\}$ (the partition with the largest possible elements), since a hybrid subgame equilibrium is not possible and the rational receiver prefers a partition with a large number of elements.

When a hybrid equilibrium is possible, we have $p \in P(b)$ and $a_0 > 0$, it is easy to see that $U_h > U_p$ and the refinement gives us the hybrid subgame equilibrium.

So when applying this refinement on the continuation game, the sender's payoff of choosing p that cannot support a hybrid equilibrium is:

$$\sum_{i=1}^{N_p}\int_{a_{i-1}}^{a_i} -(\frac{a_{i-1}+a_i}{2}-s-b)^2 ds - c(p)$$
$$= -(b^2U+\frac{U^3}{12N_p^2}+\frac{b^2U(N_p+1)(N_p-1)}{3}) - c(p)$$

which is the payoff in the finest partitional equilibrium.

The sender's payoff of choosing p that can support a hybrid equilibrium is:

$$\int_0^{a_0} -\frac{1-p}{p}b^2 ds + \sum_{i=1}^{N_h}\int_{a_{i-1}}^{a_i}\Big[-(1-p)(\frac{a_{i-1}+a_i}{2}-s-b)^2 - p(U-s-b)^2\Big]ds - c(p)$$
$$= -\frac{1-p}{p}b^2\frac{a_0}{U} - (1-p)(b^2(U-a_0)+\frac{(U-a_0)^3}{12N_h^2}+\frac{1}{3}b^2(U-a_0)(N_h+1)(N_h-1))$$
$$-p(\frac{b^3}{3}+\frac{(U-a_0-b)^3}{3}) - c(p)$$

It is trival that $p=0$ cannot support a hybrid equilibrium and it will give the sender the highest payoff among probabilities that cannot support a hybrid equilibrium. So, we just need to compare this with the highest payoff from probabilities that can support a hybrid equilibrium. If the former is larger, then the sender will choose $p=0$, otherwise the sender will choose the optimal probability among probabilities that can support a hybrid equilibrium.

When the unique equilibrium is partitional, its construction is:

The sender chooses the probability $p^*=0$ and the message $\sigma(s)=\frac{a_{i-1}+a_i}{2}$ if $s\in[a_{i-1},a_i)$

The rational receiver's on-path action is $\alpha(m,p)=m,\ m=m_i$ & $p=p^*$, and her off-path belief is $\mu(m,p)=\frac{\frac{U}{2}-pm}{1-p},\ m\neq m_i\ or\ p\neq p^*$

Here $a_i=\frac{iU}{N_p}-2i(N_p-i)b$, $i=0,1,2,...,N$,

$N_p=\max\{n\in\mathbb{N}:U>2n(n-1)b\}$

$N_h=\max\{n\in\mathbb{N}:p^*<\frac{1}{2n(n-1)}\}$.

When the unique equilibrium is hybrid, its construction is:

The sender chooses probability $p^*=\arg\max_{p\in P(b)}\Big\{-\frac{1-p}{p}b^2\frac{a_0}{U}-(1-p)(b^2(U-a_0)+$

$\frac{(U-a_0)^3}{12N_h^2} + \frac{1}{3}b^2(U-a_0)(N_h+1)(N_h-1)) - p(\frac{b^3}{3} + \frac{(U-a_0-b)^3}{3}) - c(p)\Big\}$

and the message

$$\sigma(s) = \begin{cases} s + \dfrac{b}{p^*}, & s \in [0, a_0) \\ m_i, & s \in [a_{i-1}, a_i) \ (i = 1, 2, ...N) \end{cases}$$

The rational receiver's on-path action is

$$\alpha(m,p) = \begin{cases} m - \dfrac{b}{p}, & m \in [\dfrac{b}{p^*}, a_0 + \dfrac{b}{p^*}), p = p^* \\ \dfrac{a_{i-1} + a_i}{2}, & m = m_i, p = p^* \end{cases}$$

and her off-path belief is $\mu(m,p) = \frac{\frac{U}{2} - pm}{1-p}, p \neq p^*$

and $\mu(m,p) = \begin{cases} U, & m < \dfrac{b}{p^*}, p = p^* \\ 0, & m \geqslant a_0 + \dfrac{b}{p^*} \ \& \ m \neq m_i, p = p^* \end{cases}$

Here $m_i \neq m_j$ for $i \neq j$ and $m_i \approx U$, and approximately

$a_i = a_0 + (U - a_0)\frac{i}{N_h} - 2i(N_h - i)b$, $a_0 = U - b\left(\frac{1+p^*+\sqrt{(1+p^*)^2 - 4(p^* + \frac{1-p^*}{4N_h^2})(p^* + (N_h^2 - \frac{1}{p^*})(1-p^*))}}{2(p^* + \frac{1-p^*}{4N_h^2})})\right)$,

$N_p = \max\{n \in \mathbb{N} : U > 2n(n-1)b\}$

$N_h = \max\{n \in \mathbb{N} : p^* < \frac{1}{2n(n-1)}\}$

A.6 Proposition 5

(i) When $\mathcal{S} = [0, U]$ and the distribution is uniform. Here the hybrid equilibrium cannot exist if b is very small.

If $p = 0$ is chosen, the interaction later is just like the cheap talk in Crawford and Sobel (1982), which cannot support a fully separating section with a positive measure. So, $p = 0$ cannot be chosen in a hybrid equilibrium.

Consider a candidate equilibrium with $p \geqslant \frac{1}{4}$. Note that after choosing $p > 0$, the interaction is exactly the same as the game in Ottaviani and Squintani (2006). By their result, when $p \geqslant \frac{1}{4}$, the possible hybrid strategy is $m(s) = s + \frac{b}{p}$ if $s < a_0$ and $m(s) = U$ if $s \geqslant a_0$ (only one element in the partition). If b is small enough that $a_0 > b$ for any $p \geqslant \frac{1}{4}$, consider the deviation to $p = 0$ and the message same as before for $s \in [a_0 - b, U]$, and $m \in m(.|s + b)$ for $s \in [0, a_0 - b)$.

For $s \in [0, a_0 - b)$, the difference between the sender's payoffs before and after the deviation is $U^D(s) - U(s) = 0 - (-\frac{1-p}{p}b^2 - c(p)) = \frac{1-p}{p}b^2 + c(p)$.

For $s \in [a_0 - b, a_0)$, $U^D(s) - U(s) = -b^2 - (-\frac{1-p}{p}b^2 - c(p)) = -b^2 + \frac{1-p}{p}b^2 + c(p)$.

For $s \in [a_0, U]$, $U^D(s) - U(s) \geqslant -(\frac{U+a_0}{2} - U - b)^2 - 0 = -(\frac{U-a_0}{2} + b)^2$.

We have $a_0 = U - bK(p)$ and $K(p)$ just depends on p and is bounded for $p \geqslant \frac{1}{4}$[9], and I note the upper bound of $K(p)$ when $p \geqslant \frac{1}{4}$ as $\bar{K}$. So, the difference in the ex-ante payoff is

$$\begin{aligned} U^D - U &\geqslant (\frac{1-p}{p}b^2 + c(p))\frac{a_0 - b}{U} + (-b^2 + \frac{1-p}{p}b^2 + c(p))\frac{b}{U} + (-(\frac{U-a_0}{2} + b)^2)\frac{U-a_0}{U} \\ &\geqslant (\frac{1-\bar{p}}{\bar{p}}b^2 + c(\frac{1}{4}))\frac{U - \bar{K}b - b}{U} + (-b^2 + \frac{1-\bar{p}}{\bar{p}}b^2 + c(\frac{1}{4}))\frac{b}{U} + (-(\frac{\bar{K}b}{2} + b)^2)\frac{\bar{K}b}{U} \\ &\rightarrow c(\frac{1}{4}) > 0 \end{aligned}$$

as $b \rightarrow 0$. As a result, when b is small enough, for any $p \geqslant \frac{1}{4}$, there will be a profitable deviation in p. So, when b is small enough, such p is not an equilibrium choice for the sender even if b is small enough to ensure no profitable deviation in m after choosing p.

Then consider a candidate equilibrium with $0 < p < \frac{1}{4}$. The possible hybrid strategy for this p is: $m(s) = s + \frac{b}{p}$ if $s < a_0$ and $m(s) = m_i \approx U$ if $s \in [a_{i-1}, a_i)$ $(i = 1, 2, ..., N)$. Here $p < \frac{1}{4}$, so the partition in the clustering part has at least 2 parts $(N \geqslant 2)$[10] and $U - a_{N-1} \geqslant 4b$. Again, even if b is small enough to ensures no profitable deviation in m after choosing p, consider a deviation to $p = 0$ and the same message choice as before for $s \in [0, U]$:

For $s \in [\frac{a_{N-1}+U}{2} - 2b, U]$, $U^D(s) = -(\frac{a_{N-1}+U}{2} - s - b)^2$ and $U(s) = -p(U - s - b)^2 - (1 - p)(\frac{a_{N-1}+U}{2} - s - b)^2 - c(p)$. We have $\int_{\frac{U+a_{N-1}}{2}-2b}^{U} [U^D(s) - U(s)]\frac{1}{U}ds > 0$.

[9] See Ottaviani and Squintani (2006) part 4.1.2

[10] See Ottaviani and Squintani (2006) part 4.1.3

For $s \in [a_0, \frac{a_{N-1}+U}{2} - 2b)$, since $s + b < \frac{a_{N-1}+U}{2} - b < \frac{a_{N-1}+U}{2} < U$, we have:

$$\begin{aligned} U^D(s) - U(s) &= -(\frac{a_{i-1} + a_i}{2} - s - b)^2 + p(U - s - b)^2 + (1-p)(\frac{a_{i-1} + a_i}{2} - s - b)^2 + c(p) \\ &\geqslant p[(U - s - b)^2 - (\frac{a_{N-1} + U}{2} - s - b)^2] + c(p) > 0 \end{aligned}$$

For $s \in [0, a_0)$, $U^D(s) - U(s) = -(s-s-b)^2 + p(s + \frac{b}{p} - s - b)^2 + (1-p)(s-s-b)^2 + c(p) = p[(\frac{b}{p} - b)^2 - b^2] + c(p) > 0$, since $p < \frac{1}{4}$.

So, the new expected payoff is larger than the on-path expected payoff for $0 < p < \frac{1}{4}$.

As a result, for b small enough, no $p \geqslant 0$ can be chosen for a hybrid equilibrium. So, the hybrid equilibrium does not exist.

(ii) In the partitional equilibrium, the state support is divided into N parts and cutoff points are noted as $0 = a_0 < a_1 < ... < a_N = U$, where $a_i = \frac{iU}{N} - 2i(N-i)b$ and $U > 2N(N-1)b$.

The sender's strategy $\sigma(m|s)$ is a uniform distribution among $[a_{i-1}, a_i)$ if $s \in [a_{i-1}, a_i)$[11], and $p = 0$.

The rational receiver's belief is $\mu(m) = \frac{a_{i-1}+a_i}{2}$ for $m \in [a_{i-1}, a_i)$ and the action is $\alpha(m) = \mu(m)$.

Strategies above serve as an equilibrium with b small enough and N large enough.

Fixing the strategy of the sender, the rational receiver's strategy is optimal.

Fixing the strategy of the rational receiver, if the sender deviates to $p > 0$, with $m \in \mathcal{S}$, the deviation payoff for $s \in [a_{i-1}, a_i)$ is

$$\begin{aligned} U^D(s) &\leqslant \max_m -p(m - s - b)^2 - (1-p)(\frac{a_{j-1} + a_j}{2} - s - b)^2 - c(p) \\ &\leqslant \max_m -(1-p)(\frac{a_{j-1} + a_j}{2} - s - b)^2 - c(p) = -(1-p)(\frac{a_{i-1} + a_i}{2} - s - b)^2 - c(p) := \bar{U}^D(s) \end{aligned}$$

[11] When $i = N$, this becomes $[a_{N-1}, U]$.

Then

$$\frac{\partial \bar{U}^D(s)}{\partial p} = (\frac{a_{i-1}+a_i}{2} - s - b)^2 - c'(p) \leqslant (\frac{a_i - a_{i-1}}{2} + b)^2 - c'(p)$$
$$\leqslant (\frac{U}{2N} + bN)^2 - c'(p) < (\frac{U}{N} + b)^2 - c'(p)$$

Since $c'(p) \geqslant c'(0)$, with b small enough and N large enough[12], $\frac{\partial \bar{U}^D(s)}{\partial p} < 0$ for any p and $s \in [a_{i-1}, a_i)$. And this leads to the result that $U^D(s) \leqslant \bar{U}^D(s) \leqslant -(\frac{a_{i-1}+a_i}{2} - s - b)^2 = U(s)$ ($U(s)$ is the on-path payoff for state s) for any $s \in [a_{i-1}, a_i)$. And this works for any i, so with $p > 0$ and $m \in \mathcal{S}$, any s has the payoff no larger than the on-path payoff. So, deviating to $p > 0$ will lead to the expected payoff no larger than the on-path expected payoff. Also, the sender with any state will not deviate in m when $p = 0$. As a result, when b small enough and N large enough, there is no profitable deviation for the sender.

[12] According to Crawford and Sobel (1982), the relationship between b and N is $U > 2N(N-1)b$, so N can be arbitrarily large when b is arbitrarily small.